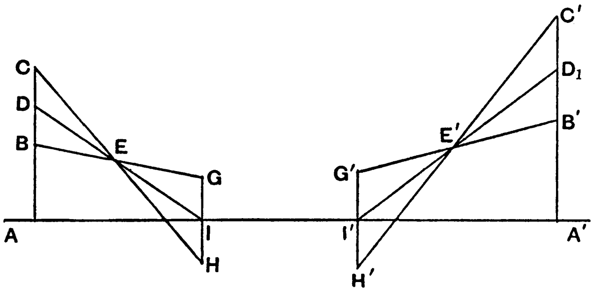
CHAPTER I

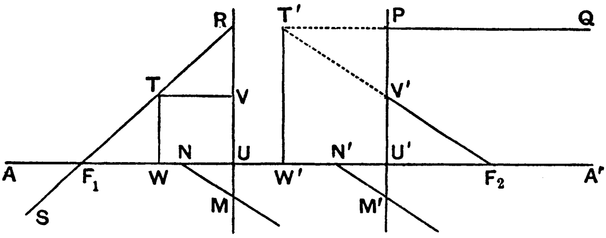
ELEMENTARY THEORY

1. *Pure Geometry.* In the absence of aberrations a symmetrical optical system may be regarded as a means of transformation between two three-dimensional regions and the transformation is such that there is a one-to-one correspondence between points, lines and planes respectively of the regions. Moreover to a straight line bisected normally by the axis of the system will correspond another straight line bisected normally by the axis, owing to the symmetry of the system.



Let BC be a line normal to the axis AA' of a symmetrical optical system and let D be its mid-point; let $B'C'$ correspond to BC and let E and E' be corresponding points. Join DE intersecting AA' in I and let BE and CE intersect a normal to the axis through I in G and H; let $H'I'G'$ correspond to HIG and join $G'E'$ and $H'E'$ passing through B' and C', since G', E', B' and H', E', C' correspond to collinear points. Let $I'E'$ intersect $B'C'$ in D_1; then D_1 corresponds to D. Since $BD=DC$, then $HI=IG$ and $H'I'=I'G'$, and therefore $B'D_1=D_1C'$; thus D_1 is the mid-point of $B'C'$. The same property may be proved for oblique lines lying in a plane normal to AA'; thus to any straight line BDC lying in a plane normal to the axis of the system and bisected at D corresponds a straight line $B'D'C'$ also lying in a normal plane and bisected at D'.

Owing to the one-to-one correspondence in points there will be two homographic ranges lying upon the axis AA' of the system; to the 'point at infinity' in the direction $A'A$ will correspond a 'flucht' point F_2—which as a special case may itself be infinitely distant; and to the 'plane at infinity' in the direction $A'A$ will correspond a normal plane through F_2. Similarly a point F_1 and a normal plane through F_1 will correspond to the 'point at infinity' and the 'plane at infinity' in the direction AA'.



Let SF_1R and RPQ be corresponding lines; then RPQ will be parallel to AA' since it must meet AA' 'at infinity'; and since to a normal plane corresponds a normal plane the locus of R will be a plane normal to the axis and intersecting it in U. Corresponding to this will be a normal plane intersecting AA' in U' and it is clear that these same two planes will be obtained by proceeding from the second region to the first region. The four points F_1, F_2, U and U' determine the transformation completely and it is to be noticed that while U and U' are corresponding points F_1 and F_2 do not correspond. Let T be any point which without loss in generality may be taken upon F_1R; to find the corresponding point draw TV parallel to AA' meeting the normal plane through U in V and let V' be the corresponding point upon the normal plane through U'. Let $V'F_2$ intersect the line PQ in T'; then T' will correspond to T. Draw TW and $T'W'$ perpendicular to AA'; then W and W' will be corresponding points. Moreover

$$F_1W \,.\, W'T' = F_1U \,.\, WT \quad \text{and} \quad W'F_2 \,.\, WT = U'F_2 \,.\, W'T',$$

so that

$$F_1W \,.\, F_2W' = -F_1U \,.\, U'F_2.$$

Let N and N' be two points upon the axis AA' such that $F_1N = U'F_2$ and $F_1U = N'F_2$; then N and N' are corresponding points. Let any line through N intersect the normal plane through U in M and let $N'M'$ be the corresponding line intersecting the normal plane through U' in M'. Then $UM = U'M'$ and $NU = N'U'$, so that NM and $N'M'$ are parallel. Thus corresponding to any straight line through N there will be a straight line through N' and these two straight lines will be parallel.

2. The transformation considered in the previous paragraph is uniquely determined when F_1 and F_2 are given in position and also one of the pairs of points U, U' and N, N'; these six points are named therefore 'cardinal points' and the normal planes through them 'cardinal planes.'

F_1 and F_2 are called 'principal foci,' U and U' 'unit points' and N and N' 'nodal' points. The following relations hold, therefore:

$$F_1U = N'F_2 = f \quad \text{and} \quad F_1N = U'F_2 = f';$$

and f and f' are named respectively the 'first' and 'second focal lengths.' No relation between f and f' can be obtained from pure geometry and recourse must be had to a law of optics which states that 'the ratio of f to f' depends only upon the optical properties of the two regions in which they are measured': this will be proved subsequently*.

In the figure of §1 let us write $F_1W = x$, $F_2W' = x'$ and $W'T' = mWT$, so that m is the 'magnification' or the ratio of the 'image' $W'T'$ to the object WT; then from the relations given there we have

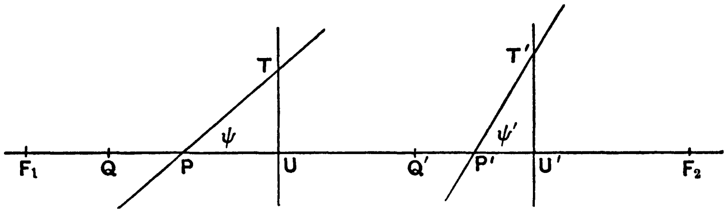
$$x = \frac{1}{m}f \quad \text{and} \quad x' = -mf',$$

so that

$$xx' = -ff'.$$

This result is due to Newton and it gives the point upon the axis in either region corresponding to a given point upon the axis in the other region and also the magnification associated with any pair of conjugate points; it is seen that the origins of co-ordinates are the principal foci and that the directions of measurement are the same.

3. Before considering formulae applicable to particular systems it may be as well to obtain a few more general results. Thus let F_1 and F_2 be the principal foci and U and U' the unit points of a symmetrical optical system



and let P, P' and Q, Q' be pairs of corresponding axial points associated respectively with magnifications m and n; and let x, x' and y, y' be the co-ordinates of these points referred to the principal foci as origins; we intend to change these origins to the conjugate points Q and Q', referred to which let the co-ordinates of P and P' be u and v respectively.

Then
$$u = QP = F_1P - F_1Q = f(1/m - 1/n)$$
and
$$v = Q'P' = F_2P' - F_2Q' = -f'(m - n)$$

* Cf. chapter II, § 3.

from the preceding paragraph; so that

$$nf'/v - f/nu = 1. \qquad \text{(1)}$$

If Q and Q' coincide with the unit points, $n = 1$, and then

$$f'/v - f/u = 1; \qquad \text{(2)}$$

if f and f' be equal we have

$$1/v - 1/u = 1/f, \qquad \text{(3)}$$

while if f and f' become infinite while their ratio remains finite according to the relation

$$f/\mu = f'/\mu',$$

μ and μ' being, so far, undefined constants, we have

$$\mu'/v - \mu/u = 0. \qquad \text{(4)}$$

This relation is of use in 'telescopic' systems.

Again, in the general case, we may define, with Maxwell, the ratio of $P'Q'$ to PQ to be the 'elongation'; and we have

$$\text{elongation} = \frac{P'Q'}{PQ} = \frac{F_2Q' - F_2P'}{F_1Q - F_1P} = \frac{(m-n)f'}{(1/n - 1/m)f} = mnf'/f. \ldots \text{(5)}$$

The longitudinal magnification at any point upon the axis is proportional therefore to the square of the transverse magnification associated with the point; and we may find similarly the oblique magnification corresponding to any oblique angle.

4. Again let PT be any line through P, intersecting the first unit plane in T and cutting the axis at an angle ψ; let $P'T'$ be the corresponding line intersecting the second unit plane at T' and cutting the axis at an angle ψ'. Then $UT = U'T'$ and

$$\therefore\ PU \tan\psi = P'U' \tan\psi',$$

i.e.,

$$f\left(1 - \frac{1}{m}\right)\tan\psi = -f'(1-m)\tan\psi',$$

$$\therefore\ f\tan\psi = mf'\tan\psi',$$

i.e.,

$$\tan\psi/\tan\psi' = mf'/f^{*}. \qquad \text{(1)}$$

If l and l' be the lengths of two corresponding normal lines at P and P' we have $l' = ml$ and then (2) takes the form

$$lf\tan\psi = l'f'\tan\psi'^{\dagger}, \qquad \text{(2)}$$

* This ratio is named, by Southall, the 'Angular Magnification' or 'Convergence-Ratio.'

† Attributed to various writers, e.g., Helmholtz, Lagrange; but it appears in Robert Smith's *Compleat Opticks*, Cambridge, 1738, for a system of thin lenses.

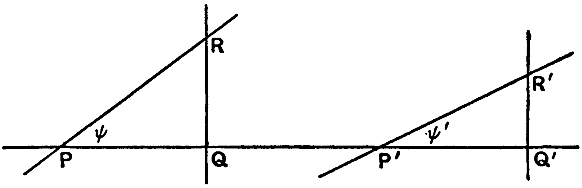
i.e., this quantity is unaltered by the transformation. If the angles involved be small we have

$$lf\psi = l'f'\psi'$$

or

$$\mu l\psi = \mu' l'\psi' \qquad \text{.............................(3)}$$

if we assume that $f/\mu = f'/\mu'$.



Let now the line through P intersect the normal plane through Q in R where $QR = y$; and let the corresponding line through P' intersect the normal plane through Q' in R' where $Q'R' = y'$. Then

$$Q'R' = ny = P'Q' \tan\psi'$$
$$= f'(m-n)\tan\psi'$$
$$= f\tan\psi - nf'\tan\psi' \qquad \text{from (1)}$$

and this is a constant quantity for rays through R'.

This may be written

$$yy' = fy\tan\psi - f'y'\tan\psi', \qquad \text{.....................(4)}$$

a symmetrical relation. It may be shewn that for lines not meeting the axis of the system their projections upon a plane passing through the axis will satisfy this relation.

The preceding results have all been obtained from a purely geometrical theory of 'collineation'—shewing how much may be derived from the mere notion of 'images'; no optical principle has been introduced so far. We shall see later that the condition that rays from a point near to P and upon the normal plane through P should be brought accurately to a focus at a point near to P' and upon the normal plane through P' is

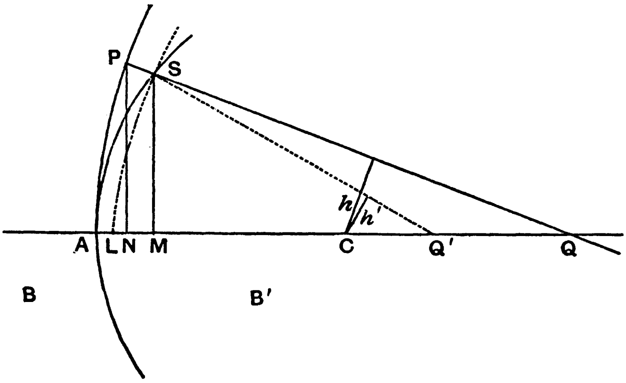
$$f\sin\psi = mf'\sin\psi', \qquad \text{.......................(5)}$$

and this will be derived from optical principles. This is clearly inconsistent with equation (2) and we draw the conclusion that a 'perfect' optical system is an impossibility*.

5. *Refraction at a Single Spherical Surface.* The preceding paragraphs have dealt with a purely geometrical transformation and we have now to make an optical application. It will be assumed for the purposes of this

* Cf. Maxwell, 'On the General Laws of Optical Instruments,' *Sci. Papers*, vol. I, pp. 271–285; Southall, *Geometrical Optics*, chap. VIII.

tract that light is propagated under the form of a wave motion and that the disturbance originating at a point of a homogeneous medium is to be found at a subsequent time upon a concentric sphere—the radius of the sphere being proportional to the time interval and depending also upon the nature of the medium.



Let P be a point upon a spherical wave-front which touches at A the spherical boundary, centre C, between two media B and B'; let a normal at P to the wave surface cut AC produced in Q and let the angle PQA be small; let this normal cut the bounding surface in S and draw NP and MS normal to AC intersecting AC in N and M respectively. After time t the disturbance from P will have reached S and that from A will have reached L, upon AC; draw the sphere with centre on AC and passing through L and S. It will be shewn that to the first approximation this sphere is the new wave-front.

Let U and R be the curvatures of the incident wave-front and of the bounding surface respectively, so that

$$U = 1/AQ \quad \text{and} \quad R = 1/AC.$$

Then if $NP = y$ we have

$$2AN = y^2 U \quad \text{and} \quad 2AM = y^2 R;$$

this is an approximation and assumes that $NP = MS$. Now the distance travelled by the light from P to the sphere is, to the same degree of approximation, NM,

i.e., $$AM - AN = \tfrac{1}{2} y^2 (R - U).$$

If v be the velocity of the luminous disturbance in the medium B,

$$2vt = y^2 (R - U);$$

and similarly if v' be the velocity in the medium B' and V the curvature of the symmetrical sphere through L and S,

$$2v't = y^2(R-V).$$

Thus $$\frac{v}{v'} = \frac{R-U}{R-V} = \frac{\mu'}{\mu} \text{ (say)}, \quad \text{.......................(1)}$$

and this relation does not involve y; thus given μ, μ', R and U there is a unique value of V, independent of t and y, i.e., of the point chosen upon the original wave-front. The symmetrical sphere through L and S is therefore the new wave-front in the medium B'.

The quantities μU and μV may be named the 'equivalent' curvatures and (1) takes the form

$$\mu' V - \mu U = (\mu' - \mu) R \quad \text{........................(2)}$$

or $$\Delta(\mu U) = (\mu' - \mu) R = \kappa \text{ (say)}, \quad \text{..................(3)}$$

where Δ is the usual operator of difference. Thus the effect of the refraction is to increase the equivalent curvature of the incident wave-front by the constant quantity $(\mu' - \mu) R$ or κ and this quantity depends only upon the properties of the media and the curvature of the bounding surface between them; it may therefore be named the 'power' of this surface.

Moreover $$\frac{\sin CSQ}{\sin CSQ'} = \frac{CQ}{SQ} \cdot \frac{SQ'}{CQ'} = \frac{R-U}{R-V} = \frac{\mu'}{\mu},$$

i.e., $$\mu' \sin CSQ' = \mu \sin CSQ,$$

to the degree of approximation contemplated.

6. If Q' be the centre of the new wave-front it will be seen that between the points Q and Q' there is a one-to-one correspondence; moreover the line of propagation PS of the incident disturbance has been transformed into the new line SQ' of propagation of the disturbance; we may employ therefore the geometrical transformation considered in the previous paragraphs. It will be observed that the results obtained are legitimate only if we neglect small quantities; the transformation given therefore is true only as a first approximation and in order to allow for this we must consider the 'aberrations' of the optical system—which accordingly will be effected in subsequent chapters.

A special case of §5 arises when the incident disturbance is reflected at the bounding surface AS—as indeed will always be the case partially. But here we may write $\mu' = -\mu$ and remember that the disturbance is to be regarded as travelling positively after incidence in the direction CA.

From (1) §5 we have $$\mu' v' = \mu v,$$

so that μ is a constant of the medium—varying indeed inversely as the velocity of the luminous disturbance in the medium; and it may be made definite by writing

$$\mu' v' = \mu v = v_0, \qquad \text{.............................(1)}$$

where v_0 is the velocity of light *in vacuo*.

A particularly simple application of (3) §5 may be considered. Consider two spherical surfaces placed close together and touching at A and let the medium between them be defined by the constant μ—the outer media being the same and defined by the constant unity. Then a double application of (3) §5 and an obvious notation leads to

$$\mu V_1 = U_1 + \kappa_1 \quad \text{and} \quad V_2 = \mu U_2 + \kappa_2;$$

where κ_1 and κ_2 are the powers of the surfaces. But $V_1 = U_2$ since the surfaces touch and therefore

$$V_2 = U_1 + \kappa_1 + \kappa_2.$$

Thus the effect of a 'thin lens' is merely to make a constant addition to the curvature of the incident wave-front.

7. *Various Formulae.* The formulae (2) and (3) §5 are the fundamental formulae for refraction at a single spherical surface; we may notice some other forms of these however which will be of use subsequently. In the first place we may change the origin of co-ordinates from the pole A to the centre of curvature C; and it is easily verified that the result is

$$\frac{1}{\mu' CQ'} - \frac{1}{\mu CQ} = \frac{\mu' - \mu}{\mu\mu'} R$$

$$= \frac{\kappa}{\mu\mu'} = J. \qquad \text{....................(1)}$$

Here J, defined by the last relation, may be named the 'modified power' of the surface; further the expression μCQ will be called the 'reduced' distance CQ, i.e., it is the geometrical distance multiplied by the constant μ of the medium in which the distance is measured. The physical meaning of this product will be examined subsequently; (1) now becomes

$$\frac{1}{CQ'} - \frac{1}{CQ} = J, \qquad \text{.........................(2)}$$

if we agree to regard all distances as reduced. Here the constant μ does not appear explicitly and this will be found to be true generally if the modified power be used instead of the ordinary power.

Again formulae involving angles are frequently of use and (2) §5 may be modified to this end. Thus if SQ and SQ' cut the axis AQQ' at angles α and α' respectively (fig. §5) we have

$$y = \alpha . AQ \quad \text{and} \quad y = \alpha' . AQ' \text{ approximately.}$$

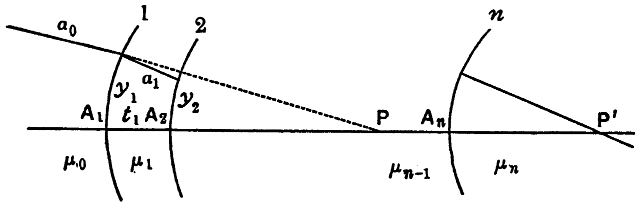
Substituting we have

$$\mu' a' - \mu a = \kappa y, \quad \ldots\ldots\ldots\ldots(3)$$

i.e.,

$$\Delta(\mu a) = \kappa y. \quad \ldots\ldots\ldots\ldots(4)$$

8. *The General System—the K-Formulae.* The symmetrical optical systemwill be composed, in general, of coaxial spherical surfaces separating various media; let us consider the optical properties of such a system.



Let n coaxial spherical surfaces, 1, ... n, separate media whose optical constants are $\mu_0, \ldots \mu_n$ and let $\kappa_1, \ldots \kappa_n$ be the powers of the surfaces; let $A_1, \ldots A_n$ be the poles of the surfaces, i.e., the points of intersection with the axis of the system. Let a ray inclined at an angle a_0 with the axis in the first medium meet the first surface in a point at distance y_1 from the axis; for the corresponding ray in the second medium let these quantities be a_1 and y_2 respectively; and so on. Let P be the intersection of the ray a_0 with the axis, P_1 that of the ray a_1, and so on; let the separations of the surfaces be t_1, t_2, ... t_{n-1} so that, e.g., $A_1 A_2 = t_1$. Then we have the following relations;

$$y_1 = A_1 P \,.\, a_0,$$
$$\mu_1 a_1 - \mu_0 a_0 = \kappa_1 y_1,$$
$$y_1 - y_2 = t_1 a_1,$$
$$\ldots\ldots\ldots\ldots$$
$$\mu_n a_n - \mu_{n-1} a_{n-1} = \kappa_n y_n;$$

which may be written

$$y_1 = \bar{u}\mu_0 a_0,$$
$$\mu_1 a_1 - \mu_0 a_0 = \kappa_1 y_1,$$
$$y_2 - y_1 = -a_1 \mu_1 a_1,$$
$$\ldots\ldots\ldots\ldots$$
$$\mu_n a_n - \mu_{n-1} a_{n-1} = \kappa_n y_n,$$

where $\bar{u}\,(\equiv A_1 P/\mu_0)$ is the 'equivalent' distance AP and $a_\lambda \mu_\lambda - t_\lambda = 0$. From this it is evident that $\mu_\lambda a_\lambda / \mu_0 a_0$ is the numerator of the 2λth convergent to the continued fraction

$$\bar{u} + \frac{1}{\kappa_1 -}\,\frac{1}{a_1 +}\,\frac{1}{\kappa_2 - \ldots +}\,\frac{1}{\kappa_n}, \quad \ldots\ldots\ldots\ldots(1)$$

while $y_\lambda/\mu_0 a_0$ is the numerator of the $(2\lambda-1)$th convergent to this continued fraction. We may denote by $K_{1,n}$ the numerator to the last convergent to the continued fraction

$$\kappa_1 - \frac{1}{a_1 +}\,\frac{1}{\kappa_1 -}\,\frac{1}{a_2 +}\,\ldots\,\frac{1}{+\,\kappa_n}.$$

Then we have the relation

$$\frac{\mu_n a_n}{\mu_0 a_0} = \bar{u} K_{1,n} + \frac{\partial K_{1,n}}{\partial \kappa_1}. \quad \ldots\ldots\ldots\ldots(2)$$

If $a_n = 0$, P will coincide with the first principal focus F_1; so that from (2)

$$F_1 A_1 = \frac{\mu_0}{K_{1,n}}\,\frac{\partial K_{1,n}}{\partial \kappa_1}, \quad \ldots\ldots\ldots\ldots(3)$$

and similarly, by starting from the other end of the system,

$$A_n F_2 = \frac{\mu_n}{K_{1,n}}\,\frac{\partial K_{1,n}}{\partial \kappa_n}. \quad \ldots\ldots\ldots\ldots(4)$$

Again P will coincide with the first nodal point if $a_0 = a_n$; so that from (2)

$$\frac{\mu_n}{\mu_0} = \frac{A_1 N}{\mu_0} K_{1,n} + \frac{\partial K_{1,n}}{\partial \kappa_1};$$

whence $F_1 N = \dfrac{\mu_n}{K_{1,n}} = f'$, the second focal length; ....(5)

similarly the first focal length is given by

$$f = \frac{\mu_0}{K_{1,n}}. \quad \ldots\ldots\ldots\ldots(6)$$

We verify therefore the general relation

$$f'/\mu_n = f/\mu_0\text{*}. \quad \ldots\ldots\ldots\ldots(7)$$

9. The quantities $K_{1,\lambda}$ and $\dfrac{\partial K_{1,\lambda}}{\partial \kappa_\lambda}$ are of importance from the point of view of subsequent aberration theory; and the following relations are given from which they can be calculated step by step. Let p_l be the numerator of the last convergent to the continued fraction

$$\kappa_1 - \frac{1}{a_1 +}\,\frac{1}{\kappa_2 -}\,\frac{1}{a_2 +}\,\ldots\,\frac{1}{+\,\kappa_\lambda};$$

then $p_l = K_{1,\lambda}$, and

$$K_{1,\lambda} = \kappa_\lambda p_{l-1} + p_{l-2}, \quad \text{so that } p_{l-1} = \frac{\partial K_{1,\lambda}}{\partial \kappa_\lambda};$$

thus

$$K_{1,\lambda} = K_{1,\lambda-1} + \kappa_\lambda \frac{\partial K_{1,\lambda}}{\partial \kappa_\lambda} \quad \ldots\ldots\ldots\ldots(1)$$

and

$$\frac{\partial K_{1,\lambda}}{\partial \kappa_\lambda} = \frac{\partial K_{1,\lambda-1}}{\partial \kappa_{\lambda-1}} - a_{\lambda-1} K_{1,\lambda-1}. \quad \ldots\ldots\ldots\ldots(2)$$

* Chap. II, § 3.