THE GENERAL THEORY OF DIRICHLET'S SERIES

I

INTRODUCTION

1. The series whose theory forms the subject of this tract are of the form

$$f(s) = \sum_{1}^{\infty} a_n e^{-\lambda_n s} \quad \text{.........................(1)},$$

where (λ_n) is a sequence of real increasing numbers whose limit is infinity, and $s = \sigma + ti$ is a complex variable whose real and imaginary parts are σ and t. Such a series is called a Dirichlet's series of *type* λ_n. If $\lambda_n = n$, then $f(s)$ is a power series in e^{-s}. If $\lambda_n = \log n$, then

$$f(s) = \sum_{1}^{\infty} a_n n^{-s} \quad \text{...........................(2)}$$

is called an *ordinary* Dirichlet's series.

Dirichlet's series were, as their name implies, first introduced into analysis by Dirichlet, primarily with a view to applications in the theory of numbers. A number of important theorems concerning them were proved by Dedekind, and incorporated by him in his later editions of Dirichlet's *Vorlesungen über Zahlentheorie*. Dirichlet and Dedekind, however, considered only real values of the variable s. The first theorems involving complex values of s are due to Jensen*, who determined the nature of the region of convergence of the general series (1); and the first attempt to construct a systematic theory of the function $f(s)$ was made by Cahen† in a memoir which, although much of the analysis which it contains is open to serious criticism, has

* Jensen, **1**, **2**. References in thick type are to the bibliography at the end of the tract.

† Cahen, **1**.

served—and possibly just for that reason—as the starting point of most of the later researches in the subject*.

It is clear that all but a finite number of the numbers λ_n must be positive. It is often convenient to suppose that they are all positive, or at any rate that $\lambda_1 \geqq 0$.†

2. It will be convenient at this point to fix certain notations which we shall regard as stereotyped throughout the tract.

(i) By $[x]$ we mean the algebraically greatest integer not greater than x. By

$$\sum_{\alpha}^{\beta} f(n)$$

we mean the sum of all values of $f(n)$ for which $\alpha \leqq n \leqq \beta$, *i.e.* for $[\alpha] \leqq n \leqq [\beta]$ or $[\alpha] < n \leqq [\beta]$, according as α is or is not an integer. We shall also write

$$A(x) = \sum_{1}^{x} a_n, \quad A(x, y) = \sum_{x}^{y} a_n, \text{‡}$$

$$\Delta a_n = a_n - a_{n+1}.$$

(ii) We shall follow Landau in his use of the symbols o, O.§ That is to say, if ϕ is a positive function of a variable which tends to a limit, we shall write

$$f = o(\phi)$$

if $f/\phi \to 0$, and

$$f = O(\phi)$$

if $|f|/\phi$ remains less than a constant K. We shall use the letter K to denote an unspecified constant, not always the same‖.

* Fuller information as to the history of the subject (up to 1909) will be found in Landau's *Handbuch der Lehre von der Verteilung der Primzahlen*, Vol. 2, Book 6, Notes and Bibliography, and in the *Encycl. des sc. math.*, T. 1, Vol. 3, pp. 249 *et seq.* We shall refer to Landau's book by the letter *H*. The two volumes are paged consecutively.

† It is evident that we can reduce the series (1) to a series satisfying this condition either (*a*) by subtracting from $f(s)$ a finite sum $\Sigma a_n e^{-\lambda_n s}$ or (*b*) by multiplying $f(s)$ by an exponential e^{-Cs}. These operations would of course change the type of the series.

‡ We shall use the corresponding notations, with letters other than a, without further explanation.

§ Landau, *H.*, p. 883, states that the symbol O seems to have been first used by Bachmann, *Analytische Zahlentheorie*, Vol. 2, p. 401.

‖ For fuller explanations see Hardy, *Orders of infinity* (Camb. Math. Tracts, No. 12), pp. 5 *et seq.*

II

ELEMENTARY THEORY OF THE CONVERGENCE OF DIRICHLET'S SERIES

1. Two fundamental lemmas. Much of our argument will be based upon the two lemmas which follow.

LEMMA 1. *We have identically*

$$\sum_x^y a_\nu \phi(\nu) = \sum_x^{y-1} A(x,n) \Delta \phi(n) + A(x,y) \phi[y].$$

This is Abel's classical lemma on partial summation*.

LEMMA 2. *If $\sigma \neq 0$, then*

$$|\Delta e^{-\lambda_n s}| \leqq \frac{|s|}{\sigma} \Delta e^{-\lambda_n \sigma}. \dagger$$

For

$$|\Delta e^{-\lambda_n s}| = \left| \int_{\lambda_n}^{\lambda_{n+1}} s e^{-us} \, du \right| \leqq |s| \int_{\lambda_n}^{\lambda_{n+1}} e^{-u\sigma} \, du = \frac{|s|}{\sigma} \Delta e^{-\lambda_n \sigma}.$$

2. Fundamental Theorems. Region of convergence, analytical character, and uniqueness of the series. We are now in a position to prove the most important theorems in the elementary theory of Dirichlet's series.

THEOREM 1. *If the series is convergent for $s = \sigma + ti$, then it is convergent for any value of s whose real part is greater than σ.*

This theorem is included in the more general and less elementary theorem which follows.

THEOREM 2. *If the series is convergent for $s = s_0$, then it is uniformly convergent throughout the angular region in the plane of s defined by the inequality*

$$|\text{am}(s - s_0)| \leqq a < \tfrac{1}{2}\pi. \ddagger$$

* Abel, **1**.

† This lemma seems to have been stated first in this form by Perron, **1**, but is contained implicitly in many earlier writings.

‡ If $s = re^{i\theta}$, we write $r = |s|$, $\theta = \text{am}\, s$. Theorem 1 is due to Jensen, **1**, and Theorem 2 to Cahen, **1**.

We may clearly suppose $s_0 = 0$ without loss of generality. We have

$$\sum_m^n a_\nu e^{-\lambda_\nu s} = \sum_m^{n-1} A(m, \nu)\, \Delta e^{-\lambda_\nu s} + A(m, n)\, e^{-\lambda_n s},$$

by Lemma 1. If ϵ is assigned we can choose m_0 so that $\lambda_m > 0$ and

$$|A(m, \nu)| < \epsilon \cos \alpha$$

for $\nu \geqq m \geqq m_0$. If now we apply Lemma 2, and observe that

$$|s|/\sigma \leq \sec \alpha$$

throughout the region which we are considering, we obtain

$$\left|\sum_m^n a_\nu e^{-\lambda_\nu s}\right| < \epsilon \left(\sum_m^{n-1} \Delta e^{-\lambda_\nu \sigma} + e^{-\lambda_n \sigma}\right) = \epsilon e^{-\lambda_m \sigma} < \epsilon$$

for $n \geqq m \geqq m_0$. Thus Theorem 2 is proved*, and Theorem 1 is an obvious corollary.

There are now three possibilities as regards the convergence of the series. It may converge for *all*, or *no*, or *some* values of s. In the last case it follows from Theorem 1, by a classical argument, that we can find a number σ_0 such that the series is convergent for $\sigma > \sigma_0$ and divergent or oscillatory for $\sigma < \sigma_0$.

THEOREM 3. *The series may be convergent for all values of s, or for none, or for some only. In the last case there is a number σ_0 such that the series is convergent for $\sigma > \sigma_0$ and divergent or oscillatory for $\sigma < \sigma_0$.*

In other words *the region of convergence is a half-plane*†. We shall call σ_0 the *abscissa of convergence*, and the line $\sigma = \sigma_0$ the *line of convergence*. It is convenient to write $\sigma_0 = -\infty$ or $\sigma_0 = \infty$ when the series is convergent for all or no values of s. *On* the line of convergence the question of the convergence of the series remains open, and requires considerations of a much more delicate character.

* It is possible to substitute for the angle considered in this theorem a wider region; *e.g.* the region

$$\sigma \geqq 0, \quad |t| \leqq e^{K\sigma} - 1$$

(Perron, **1**; Landau, *H.*, p. 739). We shall not require any wider theorem than 2. It may be added that the result of Theorem 1 remains true when we only assume that Σa_n is at most finitely oscillating: in fact, with this hypothesis, the result of Theorem 2 holds for the region

$$|\operatorname{am}(s - s_0)| \leq \alpha < \tfrac{1}{2}\pi, \quad \sigma \geqq \delta > 0,$$

as is easily proved by a trifling modification of the argument given above.

† Jensen, **1**.

3. Examples. (i) The series $\Sigma a^n n^{-s}$, where $|a| < 1$, is convergent for all values of s.

(ii) The series $\Sigma a^n n^{-s}$, where $|a| > 1$, is convergent for no values of s.

(iii) The series Σn^{-s} has $\sigma = 1$ as its line of convergence. It is not convergent at any point of the line of convergence, diverging to $+\infty$ for $s = 1$, and oscillating finitely* at all other points of the line.

(iv) The series $\sum_2^\infty (\log n)^{-2} n^{-s}$ has the same line of convergence as the last series, but is convergent (indeed absolutely convergent) at all points of the line.

(v) The series $\sum_2^\infty a_n n^{-s}$, where $a_n = (-1)^n + (\log n)^{-2}$, has the same line of convergence, and is convergent (though not absolutely) at all points of it†.

4. THEOREM 4. *Let D denote any finite region in the plane of s for all points of which*

$$\sigma \geqq \sigma_0 + \delta > \sigma_0.$$

Then the series is uniformly convergent throughout D, and its sum $f(s)$ is a branch of an analytic function, regular throughout D. Further, the series

$$\Sigma a_n \lambda_n^{\rho} e^{-\lambda_n s},$$

where ρ is any number real or complex, and λ_n^{ρ} has its principal value, is also uniformly convergent in D, and, when ρ is a positive integer, represents the function

$$(-1)^{\rho} f^{(\rho)}(s).$$

The uniform convergence of the original series follows at once from Theorem 2, since we can draw an angle of the type considered in that theorem and including D‡. The remaining results, in so far as they concern the original series and its derived series, then follow immediately from classical theorems of Weierstrass§.

When ρ is not a positive integer, we choose a positive integer m so that the real part of $\rho - m$ is negative. The series

$$\Sigma a_n \lambda_n^{\rho - m} e^{-\lambda_n s} \qquad (1)$$

may be written in the form

$$\Sigma b_n e^{-(m-\rho)\log \lambda_n} \qquad (2),$$

where $b_n = a_n e^{-\lambda_n s}$. Regarding (2) as a Dirichlet's series of type $\log \lambda_n$, and applying Theorem 1, we see that (1) is convergent whenever

* See, *e.g.*, Bromwich, **2**.

† We are indebted to Dr Bohr for this example.

‡ The vertex of the angle may be taken at σ_0, if the series is convergent for $s = \sigma_0$, and otherwise at $\sigma_0 + \eta$, where $0 < \eta < \delta$.

§ See, *e.g.*, Weierstrass, *Abhandlungen aus der Funktionentheorie*, pp. 72 *et seq.*; Osgood, *Funktionentheorie*, Vol. 1, pp. 257 *et seq.*

$\Sigma a_n e^{-\lambda_n s}$ is convergent. The proof of the theorem may now be completed by a repetition of our previous arguments.

THEOREM 5. *If the series is convergent for $s = s_0$, and has the sum $f(s_0)$, then $f(s) \to f(s_0)$ when $s \to s_0$ along any path which lies entirely within the region*

$$|\operatorname{am}(s - s_0)| \leqq \alpha < \tfrac{1}{2}\pi.$$

This theorem* is an immediate corollary from Theorem 2. It is of course only when s_0 lies on the line of convergence that it gives us any information beyond what is given by Theorem 4.

5. THEOREM 6. *Suppose that the series is convergent for $s = 0$, and let E denote the region*

$$\sigma \geqq \delta > 0, \quad |\operatorname{am} s| \leqq \alpha < \tfrac{1}{2}\pi.$$

Suppose further that $f(s) = 0$ for an infinity of values of s lying inside E. Then $a_n = 0$ for all values of n.

The function $f(s)$ cannot have an infinity of zeros in the neighbourhood of any finite point of E, since it is regular at any such point. Hence we can find an infinity of values $s_n = \sigma_n + t_n i$, where $\sigma_{n+1} > \sigma_n$, $\lim \sigma_n = \infty$, such that $f(s_n) = 0$.

But
$$g(s) = e^{\lambda_1 s} f(s) = a_1 + \sum_2^\infty a_n e^{-(\lambda_n - \lambda_1) s}$$
is convergent for $s = 0$ and so uniformly convergent in E. Hence

$$g(s) \to a_1$$

when $s \to \infty$ along any path in E. This contradicts the fact that $g(s_n) = 0$, unless $a_1 = 0$. It is evident that we may repeat this argument and so complete the proof of the theorem †.

6. Determination of the abscissa of convergence. Let us suppose that the series is not convergent for $s = 0$, and let

$$\overline{\lim} \frac{\log |A(n)|}{\lambda_n} = \gamma. \ddagger$$

* The generalisation of the 'Abel-Stolz' theorem for power series (Abel, **1**; Stolz, **1**, **2**).

† This theorem, like Theorem 2 itself, may be made wider: see Perron, **1**; Landau, *H.*, p. 745. Until recently it was an open question whether it were possible that $f(s)$ could have zeros whose real parts surpass all limit: all that Theorem 6 and its generalisations assert is that the imaginary parts of such zeros, if they exist, must increase with more than a certain rapidity. The question has however been answered affirmatively by Bohr, **4**. But if there is a region of absolute convergence, the answer is negative (see III, § 5).

‡ By $\overline{\lim}\, u_n$ we denote the 'maximum limit' of the sequence u_n: *cf.* Bromwich, *Infinite series*, p. 13.

It is evident that $\gamma \geqq 0$*. We shall now prove that $\sigma_0 = \gamma$.

(i) Let δ be any positive number. We shall prove first that the series is convergent for $s = \gamma + \delta$.

Choose ϵ so that $0 < \epsilon < \delta$. Then, by the definition of γ, we have

$$\log |A(\nu)| < (\gamma + \delta - \epsilon) \lambda_\nu, \quad |A(\nu)| < e^{(\gamma+\delta-\epsilon)\lambda_\nu}$$

for sufficiently large values of ν. Now

$$\sum_1^n a_\nu e^{-\lambda_\nu s} = \sum_1^{n-1} A(\nu) \Delta e^{-\lambda_\nu s} + A(n) e^{-\lambda_n s}.$$

The last term is, for sufficiently large values of n, less in absolute value than $e^{-\epsilon\lambda_n}$, and so tends to zero; and everything depends on establishing the convergence of the series

$$\Sigma e^{(\gamma+\delta-\epsilon)\lambda_\nu} \Delta e^{-(\gamma+\delta)\lambda_\nu}.$$

Now, since $\gamma + \delta - \epsilon$ is positive, we have

$$e^{(\gamma+\delta-\epsilon)\lambda_\nu} \Delta e^{-(\gamma+\delta)\lambda_\nu} = (\gamma+\delta) \int_{\lambda_\nu}^{\lambda_{\nu+1}} e^{(\gamma+\delta-\epsilon)\lambda_\nu - (\gamma+\delta)x} dx$$

$$< (\gamma+\delta) \int_{\lambda_\nu}^{\lambda_{\nu+1}} e^{-\epsilon x} dx;$$

and the series $$(\gamma+\delta) \Sigma \int_{\lambda_\nu}^{\lambda_{\nu+1}} e^{-\epsilon x} dx$$

is obviously convergent. It follows that

$$\sigma_0 \leqq \gamma.$$

(ii) Suppose $$\Sigma a_\nu e^{-\lambda_\nu s} = \Sigma b_\nu \qquad (s > 0)$$

convergent. Then

$$A(n) = \sum_1^n b_\nu e^{\lambda_\nu s} = \sum_1^{n-1} B(\nu) \Delta e^{\lambda_\nu s} + B(n) e^{\lambda_n s}.$$

It follows that $$|A(n)| < K e^{\lambda_n s},$$

and therefore that

$$\log |A(n)| < \lambda_n s + K < (s+\delta) \lambda_n,$$

for any positive δ, if n is large enough. Hence

$$s \geqq \overline{\lim} \frac{\log |A(n)|}{\lambda_n} = \gamma,$$

and therefore $$\sigma_0 \geqq \gamma.$$

* We can determine a constant K such that $\log |A(n)| > -K$ for an infinity of values of n. This would still be true if Σa_n converged to a sum other than zero: but if the sum were zero we should have

$$\log |A(n)| \to -\infty.$$

From the results of (i) and (ii) we deduce

THEOREM 7. *If the abscissa of convergence of the series is positive, it is given by the formula*

$$\sigma_0 = \overline{\lim} \frac{\log |A(n)|}{\lambda_n}.*$$

7. Absolute convergence of Dirichlet's series. We can apply the arguments of the preceding sections to the series

$$\Sigma |a_n| e^{-\lambda_n s} \quad \text{..............................(1).}$$

We deduce the following result:

THEOREM 8. *There is a number $\bar{\sigma}$ such that the series* (1) *is absolutely convergent if $\sigma > \bar{\sigma}$ and is not absolutely convergent if $\sigma < \bar{\sigma}$. This number, if positive, is given by the formula*

$$\bar{\sigma} = \overline{\lim} \frac{\log \bar{A}(n)}{\lambda_n},$$

where $\bar{A}(n) = |a_1| + |a_2| + \ldots + |a_n|$.

In other words a Dirichlet's series possesses, besides its abscissa, line, and half-plane of convergence, an abscissa, line, and half-plane of absolute convergence. It should however be observed that the theorem which asserts the existence of a half-plane of absolute convergence is in reality more elementary than Theorem 3, as it follows at once from the inequality

$$|e^{-\lambda_\nu s}| \leq |e^{-\lambda_\nu s_1}| \qquad (\sigma \geqq \sigma_1),$$

and does not depend on Lemma 1.

It is evident that $\bar{\sigma} \geqq \sigma_0$. We may of course have $\bar{\sigma} = \infty$ or $\bar{\sigma} = -\infty$. In general there will be a *strip* between the lines of convergence and absolute convergence, throughout which the series is conditionally convergent. This strip may vanish (if $\bar{\sigma} = \sigma_0$) or comprise the whole plane (if $\sigma_0 = -\infty$, $\bar{\sigma} = \infty$) or a half-plane (if

* Cahen, **1**. Dedekind, *l.c.* p. 1, and Jensen, **2**, had already given results which together contain the substance of the theorem. The result holds when $\sigma_0 = 0$, unless Σa_n converges to zero. If $\sigma_0 < 0$ the result is in general untrue. It is plain that in such a case we can find σ_0 by first applying to the variable s such a linear transformation as will make the abscissa of convergence positive. But there is a formula directly applicable to this case, viz.

$$\sigma_0 = \overline{\lim} \frac{\log |A - A(n)|}{\lambda_{n+1}},$$

where A is the sum of the series Σa_n (obviously convergent when $\sigma_0 < 0$). This formula was given (with a slight error, viz. λ_n for λ_{n+1}) by Pincherle, **1**: see also Knopp, **6**; Schnee, **6**. Formulae applicable in *all* cases have been found by Knopp, **6** (for the case $\lambda_n = \log n$ only); Kojima, **1**; Fujiwara, **1**; and Lindh (Mittag-Leffler, **1**).

$\sigma_0 = -\infty$, $-\infty < \bar{\sigma} < \infty$ or $-\infty < \sigma_0 < \infty$, $\bar{\sigma} = \infty$). For Dirichlet's series of a given type, however, its breadth is subject to a certain limitation.

THEOREM 9. *We have* $\bar{\sigma} - \sigma_0 \leqq \overline{\lim} \dfrac{\log n}{\lambda_n}$.

We shall prove this theorem on the assumption that $\sigma_0 > 0$; its truth is obviously independent of this restriction. Given δ, we can choose n_0 so that

$$|A(n)| < e^{(\sigma_0+\delta)\lambda_n} \qquad (n \geqq n_0),$$

and accordingly

$$|a_n| = |A(n) - A(n-1)| < 2e^{(\sigma_0+\delta)\lambda_n} < e^{(\sigma_0+2\delta)\lambda_n}.*$$

Hence $$\bar{A}(n) = \sum_1^n |a_\nu| < \bar{A}(n_0) + n e^{(\sigma_0+2\delta)\lambda_n} < n e^{(\sigma_0+3\delta)\lambda_n}$$

if $n \geqq n_1$ and n_1 is sufficiently large in comparison with n_0. Thus

$$\frac{\log \bar{A}(n)}{\lambda_n} < \frac{\log n}{\lambda_n} + \sigma_0 + 3\delta \qquad (n \geqq n_1),$$

from which the theorem follows immediately.

If $\log n = o(\lambda_n)$, the lines of convergence and absolute convergence coincide: in particular this is the case if $\lambda_n = n$. In this case our theorems become, on effecting the transformation $e^{-s} = x$, classical theorems in the theory of power series. Thus Theorems 1 and 3 establish the existence of the circle of convergence, and 7 gives a slightly modified form of Cauchy's formula for the radius of convergence. Theorems 2, 4, 5, and 6 also become familiar results. If $\lambda_n = \log n$, the maximum possible distance between the lines of convergence is 1. This is of course an obvious consequence of the fact that $\Sigma n^{-1-\delta}$ is convergent for all positive values of δ.

It is not difficult to construct examples to show that every logically possible disposition of the lines of convergence and absolute convergence, consistent with Theorem 9, may actually occur. We content ourselves with mentioning the series

$$\Sigma \frac{(-1)^n}{\sqrt{n}} (\log n)^{-s},$$

which is convergent for all values of s, but never absolutely convergent.

8. It will be well at this point to call attention to the essential difference which distinguishes the general theory of Dirichlet's series from the simpler theory of power series, and lies at the root of the particular difficulties of the former. The region of convergence of a power series is determined in the simplest possible manner by the disposition of the singular points of the function which it represents: the circle of convergence extends up to the nearest singular point. As we shall see, no such simple relation holds in the general case; a Dirichlet's series convergent in a portion of the plane only may represent a function regular all over the plane, or in a wider region of

* If $e^{\delta\lambda_n} > 2$ for $n \geqq n_0$, as we can obviously suppose.

it. The result is (to put it roughly) that many of the peculiar difficulties which attend the study of power series *on* the circle of convergence are extended, in the case of Dirichlet's series, to wide regions of the plane or even to the whole of it.

There is however one important case in which the line of convergence necessarily contains at least one singularity.

THEOREM 10. *If all the coefficients of the series are positive or zero, then the real point of the line of convergence is a singular point of the function represented by the series**.

We may suppose that $\sigma_0 = \bar{\sigma} = 0$. Then, if $s = 0$ is a regular point, the Taylor's series for $f(s)$, at the point $s = 1$, has a radius of convergence greater than 1. Hence we can find a negative value of s for which

$$f(s) = \sum_{\nu=0}^{\infty} \frac{(s-1)^\nu}{\nu!} f^{(\nu)}(1) = \sum_{\nu=0}^{\infty} \frac{(1-s)^\nu}{\nu!} \sum_{n=1}^{\infty} a_n \lambda_n^\nu e^{-\lambda_n}.$$

But every term in this repeated series is positive. Hence† the order of summation may be inverted, and we obtain

$$f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n} \sum_{\nu=0}^{\infty} \frac{(1-s)^\nu \lambda_n^\nu}{\nu!} = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}.$$

Thus the series is convergent for some negative values of s, which contradicts our hypotheses.

In the general case all conceivable hypotheses may actually be realised. Thus the series

$$1^{-s} - 2^{-s} + 3^{-s} - \dots,$$

which converges for $\sigma > 0$, represents the function

$$(1 - 2^{1-s}) \zeta(s), \ddagger$$

which is regular all over the plane. The series

$$\Sigma 2^{-2^n s}$$

has the imaginary axis as a line of essential singularities§.

* This theorem was proved first for power series by Vivanti, **1**, and Pringsheim, **1**. It was extended to the general case by Landau, **1**, and *H.*, p. 880. Further interesting generalisations have been made by Fekete, **1**, **2**.

† Bromwich, *Infinite series*, p. 78.

‡ For the theory of the famous ζ-function of Riemann, we must refer to Landau's *Handbuch* and the *Cambridge Tract* by Messrs Bohr and Littlewood which, we hope, is to follow this.

§ Landau, **2**. General classes of such series have been defined by Knopp **4**. Schnee, **1**, **3**, and Knopp, **1**, **3**, **5**, have also given a number of interesting theorems relating to the behaviour of $f(s)$ as s approaches a singular point on the line of convergence, the coefficients of the series being supposed to obey certain asymptotic laws. These theorems constitute a generalisation of the work of Appell, Cesàro, Lasker, Pringsheim and others on power series.