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# A LOCUS
# WITH 25920 LINEAR SELF-TRANSFORMATIONS

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# CONTENTS

# PREFACE

THIS VOLUME is concerned with a locus—itself very interesting to explore geometrically—which exhibits in a simple way the structure of the group of the lines of a cubic surface in ordinary space, regarded as the group of the tritangent planes of the surface. Incidentally certain quite elementary results for the substitutions of 4, 5 and 6 objects are necessary; and, for the sake of completeness, these are explained in detail. Historically the linear expression for the group of transformations considered, and of the different linear expression briefly sketched in Note 2, of the Appendix, both arose from the theory of the linear transformations of the periods of theta functions of two variables; but, beyond references to the literature, this is not dealt with. It is hoped that the Introduction to the text, and the list of headings of the sections, will make sufficiently clear what is included. A brief index of notations is also appended. The argument of the text requires frequent reference to the Scheme of synthemes given as frontispiece of the volume.

To some it may seem that such a theme—at this time—is futile. It is possible, however, to take the view that the primary purpose of the pursuit of science is not the advancement of technology, but the widening of the horizon of the human mind. In this mathematics has always borne an honourable, often a decisive part; indeed, many cases could be cited to support the more extreme view that the development of mathematical ideas, and the emergence of new physical conceptions, are intimately related.

The general theory of linear groups has developed widely on the algebraic side since the group considered here—one of the earliest—was established; and even of this the present is a very incomplete account. The writer has had the advantage of the co-operation, in the reading of the proof-sheets, of Dr J. A. Todd, who has himself written on the matter, and is very greatly indebted to him. He would wish also to acknowledge his obligation to the Staff of the University Press, especially at this time of difficulty in the printing of books.

H. F. B.

26 *November* 1945

# INTRODUCTION

In his monumental volume on the theory of substitutions (*Traité des substitutions*, Paris, 1870), Jordan considers the group of the lines of a cubic surface in ordinary space, which he regards primarily as the group of the substitutions of the tritangent planes of the surface. Later in the same volume, with acknowledgements to Kronecker, he considers the group of the trisection of the periods of a theta function of two variables, proving that the study of this group is essentially the same problem as that of the group of the lines of a cubic surface. In a series of papers* on hyperelliptic functions of two variables, in the *Math. Ann.*, Burkhardt obtains five theta functions which are linearly transformed among themselves by the group of the trisection, thus incidentally obtaining for the first time the expression of the group of the lines of a cubic surface by linear equations (which arises also in a different form in his fourth memoir); and he investigates the homogeneous polynomials in these five functions which are invariant under the resulting linear group. The simplest of these invariants is of the fourth order in the five functions. When equated to zero, this represents a primal in space of four dimensions, which, considering the thoroughness of Burkhardt's work in the four memoirs quoted, may be described as *Burkhardt's Primal*. The geometrical properties of this primal are very interesting; and they form a vivid and simple concrete representation of the group of the lines of a cubic surface, and its more important subgroups; and incidentally illustrate the elements of the theory

* Burkhardt, *Math. Ann.* XXXV (1889), pp. 198–296; XXXVI (1889), pp. 371–434; XXXVIII (1890), pp. 161–224, 309–12; XLI (1892), pp. 313–43. It is the third of these memoirs XXXVIII (1890), which mainly concerns us here, but reference is made to the fourth memoir in Note 2 of the Appendix at the end of this volume.

of the substitutions of five and six objects. After Burkhardt there are two interesting papers by A. B. Coble (*American J. Math.*, XXVIII (1906), pp. 333–66, and *Trans. Amer. Math. Soc.* XVIII (1917), pp. 331–72), in which the geometrical properties of the primal are considered. Coble gives explicitly a symmetrical form of the equation of the primal (to which the equations of transformation are given by Burkhardt, *Math. Ann.* XXVIII, p. 205). On account of its symmetry this form is adopted here as fundamental. Still later Dr J. A. Todd (*Quart. J. Math.* VII (1936), pp. 168–74) has added to his other papers on quartic primals in four dimensions a masterly proof that Burkhardt's primal is rational (that is, that its four independent coordinates are expressible rationally in terms of three rational functions of themselves), without, however, obtaining the explicit reverse equations. It is remarked here that this rationality is obvious when it is seen that there exist on the primal (many) sets of three planes of which every two have only a point in common; and the reverse equations are obtained in one of the possible 72.40 ways.

The present account is primarily a study of the geometrical properties of the primal; and, to be intelligible, must needs contain many results that are not novel. But there are two features which, so far as I know, are new. The first is a notation for the forty-five nodes of the primal (and, therefore, effectively, for the tritangent planes of a cubic surface) which enables the relations of these nodes to be simply described and verified. The second is the reference to the projections into itself of which the primal is capable, of which I have seen no mention. All Burkhardt's fundamental transformations are expressed here in terms of these projections. Burkhardt's proof that these fundamental transformations generate the group depends upon their derivation from linear transformation of the periods of the hyperelliptic functions, and so belongs to the theory of linear transformation of the periods. It would seem that what is advanced below in regard to the geometrical projections is sufficient to enable us to dispense with reference to these periods; but a formal proof of this requires further elaboration. The elementary results which arise for the substitutions of five and six objects will not be new to those who have studied the theory of groups of substitutions, but may be welcome, from their concrete character, to those less familiar with the theory.

One remark should perhaps be added here to make the general statements of this introduction more precise: The group of the lines of a cubic surface is of order $2^4 . 3^4 . 40$; this group has a subgroup of order $\frac{1}{2}(2^4 . 3^4 . 40)$ or $2^3 . 3^4 . 40$, which, as Jordan proved, is *simple*. This subgroup, regarded, as by Jordan, as a group of substitutions of the tritangent planes, contains only *even* substitutions of these. It is this subgroup which is considered here.