# A LOCUS WITH 25920 LINEAR SELF-TRANSFORMATIONS

(1) **The fundamental notation.** In a space of four dimensions we may use for homogeneous coordinates the six

$$x_1, x_2, x_3, x_4, x_5, x_6,$$

connected by the equation $x_1+x_2+\ldots+x_6=0$. Throughout we shall use $\epsilon$ for the cube-root of unity, $\exp(2\pi i/3)$. If $l, m, n, p, q, r$ denote the numbers 1, 2, 3, 4, 5, 6, in any order, the symbol $(lp\,.\,mq\,.\,nr)$ shall denote the point for which $x_l=x_p=1$, $x_m=x_q=\epsilon$, $x_n=x_r=\epsilon^2$. The same point is then represented by the symbol $(mq\,.\,nr\,.\,lp)$, or by the symbol $(nr\,.\,lp\,.\,mq)$. Occasionally, a couple such as $l, p$ may be spoken of as a *duad*, and the symbol $(lp\,.\,mq\,.\,nr)$, whose three duads contain all the numbers 1, 2, ..., 6; may be spoken of as a *syntheme* (after Sylvester). Similarly the symbol $[lp\,.\,mq\,.\,nr]$ shall denote the linear function of the coordinates $x_l+x_p+\epsilon(x_m+x_q)+\epsilon^2(x_n+x_r)$; the prime represented by $[lp\,.\,mq\,.\,nr]=0$ is the same as either of those denoted by $[mq\,.\,nr\,.\,lp]=0$, or $[nr\,.\,lp\,.\,mq]$. Further $(lp)$ shall denote the point whose coordinates are $x_l=1$, $x_p=-1$, with $x_m=x_q=x_n=x_r=0$, and $[lp]$ shall denote the linear function $x_l-x_p$.

We consider the square scheme of synthemes which, for facility of reference, is printed as frontispiece of the volume, of which the columns are denoted respectively by $\{A\}$, $\{B\}$, $\{C\}$, $\{D\}$, $\{E\}$, $\{F\}$ and the rows by $\{A_0\}$, $\{B_0\}$, $\{C_0\}$, $\{D_0\}$, $\{E_0\}$, $\{F_0\}$. In each column, and in each row, all the fifteen duads of two numbers from 1, 2, 3, 4, 5, 6 occur, each once; the thirty synthemes which occur are all different, but to a syntheme occurring in any row and column there corresponds a syntheme, occurring in the same column and row, differing from the former syntheme by the interchange of the order of the second and third duads. Such a scheme is referred to by Sylvester (*Coll. Papers*, I (1844), p. 92, and *Coll. Papers*, II (1861), p. 265), and may be used in connexion with the theory of the Pascal lines of six points of a conic (Baker, *Principles of Geometry* II (1930), p. 221). But in both these cases the order of the duads in any syntheme is indifferent, while here this order is of the essence of the notation. In this scheme, any duad occurs once in any row, and once in any column; and any two duads that occur once together occur also together in another syntheme, but in reverse order. The scheme thus represents thirty points, each of which can also be characterized by the row and column in which it appears, and denoted by a symbol $(PQ_0)$, or $(Q_0P)$, where $P$ is one of $A, B, \ldots, F$, and $Q_0$ is one of $A_0, B_0, \ldots, F_0$. For instance (14.36.25) may be denoted by $(AB_0)$. It *will be convenient then to denote* [14.36.25] by $[AB_0]$; *and so in general.*

It may then be verified that in any column, as in any row, the five points represented by the synthemes form a *simplex* in the space of four dimensions, any four of these defining a prime (or space of three dimensions). In fact, the points in the first column, other than the first of these, (14.36.25), or $(AB_0)$, define the prime $[14.25.36] = 0$, or $[A_0B] = 0$; and the points in the first row, other than the first of these, (14.25.36), or $(A_0B)$, define the prime $[14.36.25] = 0$, or $[AB_0] = 0$. Or, generally, if $P$, $Q$ denote two of the letters $A, B, \ldots, F$, the points of the column $\{P\}$ other than $(PQ_0)$ determine the prime $[P_0Q] = 0$; and the points of the row $\{Q_0\}$ other than $(PQ_0)$ equally lie in this prime $[P_0Q] = 0$. (This prime, $[P_0Q] = 0$, thus contains eight points, and we shall see that it also contains the point $(P_0Q)$, as well as the three points $(lp)$, $(mq)$, $(nr)$, if $(PQ_0) = (lp.mq.nr)$.) As we have said, in all cases the synthemes $(P_0Q)$, or $[P_0Q]$, are obtained respectively from $(PQ_0)$, or $[PQ_0]$, by interchange of the second and third duads of the syntheme.

We shall speak of the simplex, whose angular points are those of the column $\{P\}$, as a *pentahedron* $\{P\}$, and equally of the simplex, whose angular points are those of the row $\{Q_0\}$, as the pentahedron $\{Q_0\}$. Thus the pentahedron $\{A\}$ has, for its angular points, the points $(AB_0), (AC_0), \ldots, (AF_0)$, and has for prime faces respectively opposite to these the primes $[A_0 B] = 0, [A_0 C] = 0, \ldots, [A_0 F] = 0$. Similarly the pentahedron $\{B_0\}$ has for its angular points $(AB_0)$, $(CB_0)$, $(DB_0)$, ..., $(FB_0)$, and for opposite prime faces respectively $[A_0 B] = 0$, $[C_0 B] = 0$, ..., $[F_0 B] = 0$. We shall also speak of the twelve pentahedra so arising as *Jordan pentahedra*, since the angular points of any one of these essentially occur, associated together, in Jordan's theory of the group of the lines of a cubic surface.

We shall also consider, however, fifteen other Jordan pentahedra. Suppose that the syntheme $(PQ_0)$ is $(lp.mq.nr)$, so that $(P_0 Q)$ is $(lp.nr.mq)$. It can then be verified that the five points $(PQ_0)$, $(P_0 Q)$, $(lp)$, $(mq)$, $(nr)$ form a simplex, with prime faces respectively opposite to these given by $[P_0 Q] = 0$, $[PQ_0] = 0$, $[lp] = 0$, $[mq] = 0$, $[nr] = 0$. This simplex we speak of as the pentahedron $\{PQ\}$. In addition then to the forty-five points $(PQ_0)$, $(lp)$, we consider, in all, twenty-seven pentahedra, $\{P\}$, $\{Q_0\}$, $\{PQ\}$. An angular point of one of these pentahedra will be called the *pole* of the opposite prime face of this pentahedron, this being spoken of as the *polar prime* of the opposite angular point. If $\xi_1, \xi_2, \ldots, \xi_6$ be the coordinates of one of the forty-five points, and the equation of its polar prime be written

$$u_1 x_1 + u_2 x_2 + \ldots + u_6 x_6 = 0,$$

where the indefiniteness of $u_1, \ldots, u_6$ which arises from

$$x_1 + x_2 + \ldots + x_6 = 0$$

is removed by making the condition $u_1 + u_2 + \ldots + u_6 = 0$, then, in every case, $u_i$ is the conjugate imaginary of $\xi_i$, or, say $u_i = \bar{\xi}_i$. If one of the forty-five primes, say the polar of $(\xi_1, \xi_2, \ldots, \xi_6)$, contain a particular one, $(\eta_1, \eta_2, \ldots, \eta_6)$, of the forty-five points, then the polar prime of this point contains the pole of the prime in question; for if $\bar{\xi}_1 x_1 + \ldots + \bar{\xi}_6 x_6 = 0$ contain $(\eta_1, \ldots, \eta_6)$, then

$$\bar{\xi}_1 \eta_1 + \ldots + \bar{\xi}_6 \eta_6 = 0,$$

and this involves $\bar{\eta}_1\xi_1 + \ldots + \bar{\eta}_6\xi_6 = 0$, where $\bar{\xi}_i$ denotes the conjugate imaginary of $\xi_i$, etc. Each of the forty-five points is an angular point of three of the pentahedra; for instance $(AB_0)$ is an angular point of the three pentahedra $\{A\}$, $\{B_0\}$, $\{AB\}$; and (14) is an angular point of $\{AB\}$, $\{CD\}$ and $\{EF\}$, since, as the fundamental scheme given above shews, the duad 14 occurs in the synthemes $(AB_0)$ or $(A_0B)$, $(CD_0)$ or $(C_0D)$, $(EF_0)$ or $(E_0F)$; and we have in fact $3.45 = 5.27$. Dually, the opposite prime faces of the three pentahedra which have a common angular point coincide in the polar prime of this point, which thus contains the twelve angular points of these three pentahedra other than their common angular point. For instance, if the angular point be $(AB_0)$, the twelve points $(AC_0)$, $(AD_0)$, $(AE_0)$, $(AF_0)$; $(B_0C)$, $(B_0D)$, $(B_0E)$, $(B_0F)$; $(A_0B)$, (14), (36), (25), all lie in $[A_0\dot{B}] = 0$; or again, if the angular point be (14), the twelve points $(AB_0)$, $(A_0B)$, (36), (25); $(CD_0)$, $(C_0D)$, (23), (56); $(EF_0)$, $(E_0F)$, (26), (35), which can be written down by the fundamental scheme, since $(AB_0) = (14.36.25)$, $(CD_0) = (14.23.56)$, $(EF_0) = (14.26.35)$, all lie in the prime $[14] = 0$. Algebraically, if $(\xi_1, \ldots, \xi_6)$, $(\eta_1, \ldots, \eta_6)$ be two angular points of the same pentahedron, we have

$$\bar{\xi}_1\eta_1 + \ldots + \bar{\xi}_6\eta_6 = 0.$$

Dually, each of the forty-five primes contains three sets of four, of the forty-five points, each set consisting of the angular points of a pentahedron whose other angular point is the pole of the prime. Any one of these forty-five primes may be spoken of as a *Jordan prime*.

(2) **The equation of the Burkhardt primal.** Now consider the locus, in the space of four dimensions in which the coordinates are $x_1, x_2, \ldots, x_6$, subject to $x_1 + x_2 + \ldots + x_6 = 0$, which is represented by the equation $f = 0$, where

$$f = \Sigma x_l x_m x_n x_p$$

is the sum of the fifteen products of four different coordinates. As has been said, this explicit equation is given by Coble (*loc. cit.*), but such coordinates as $x_1, \ldots, x_6$ were suggested by Burkhardt, and their expression given in terms of the coordinates he employed (*Math. Ann.* xxxviii, p. 205).

Then it is easy to verify that the forty-five points, consisting of the thirty points $(PQ_0)$, and the fifteen points $(lp)$, are all conical nodes of the primal. For the equation $f = 0$ is of the form

$$(x_l + x_m + x_n)\, x_p x_q x_r + (x_p + x_q + x_r)\, x_l x_m x_n + (x_m x_n + x_n x_l + x_l x_m)(x_q x_r + x_r x_p + x_p x_q) = 0,$$

where $l, m, n, p, q, r$ denote $1, 2, \ldots, 6$ in any order. And the conditions for a node (in virtue of the relation $x_1 + \ldots + x_6 = 0$) are that all the first derivatives $\partial f/\partial x_i$ should be equal; also

$$\partial f/\partial x_l = x_p x_q x_r + (x_p + x_q + x_r)\, x_m x_n + (x_m + x_n)(x_q x_r + x_r x_p + x_p x_q),$$

$$\partial f/\partial x_p = (x_l + x_m + x_n)\, x_q x_r + x_l x_m x_n + (x_m x_n + x_n x_l + x_l x_m)(x_q + x_r);$$

thus, at the point $(lp.mq.nr)$, for which

$$(x_l, x_m, x_n, x_p, x_q, x_r) = (1, \epsilon, \epsilon^2, 1, \epsilon, \epsilon^2),$$

since

$$x_p x_q x_r = 1, \quad x_p + x_q + x_r = 0, \quad x_q x_r + x_r x_p + x_p x_q = 0,$$
$$x_l + x_m + x_n = 0, \quad x_l x_m x_n = 1, \quad x_m x_n + x_n x_l + x_l x_m = 0,$$

we see that all the six derivatives are equal to 1; and at the point $(lp)$, for which $x_l = 1$, $x_p = -1$, $x_m = x_q = x_n = x_r = 0$, all the six derivatives vanish.

Conversely it is easy to verify, from a more compendious form of the equation of the primal in terms of five homogeneous variables, which occurs below (§ (13)), that the primal has no other nodes than these forty-five.

(3) **Similarity, or equal standing, of the forty-five nodes, and of the twenty-seven pentahedra.** It is clear, from the symmetry of the equation of the primal, that the thirty nodes $(lp.mq.nr)$ are entirely similar to one another, and the fifteen nodes $(lp)$ are likewise similar to one another. Likewise that the

twelve pentahedra $\{P\}$, $\{Q_0\}$ are similar to one another, and the fifteen pentahedra $\{PQ\}$ are similar to one another. In fact, *any* two of the nodes, and *any* two of the pentahedra, are similar to one another, notwithstanding the difference of notation. This will appear at once if we put down a linear transformation of the coordinates which leaves the equation of the primal unaltered and changes any chosen one of the fifteen pentahedra $\{PQ\}$, say the pentahedron $\{AB\}$, into one of the twelve pentahedra $\{P\}$, $\{Q_0\}$. Many such transformations are possible; we choose that given by

$$x_1' = x_1 + \epsilon x_2 + \epsilon x_3, \quad -x_4' = x_4 + \epsilon^2 x_5 + \epsilon^2 x_6,$$
$$x_2' = \epsilon x_1 + x_2 + \epsilon x_3, \quad -x_5' = \epsilon^2 x_4 + x_5 + \epsilon^2 x_6,$$
$$x_3' = \epsilon x_1 + \epsilon x_2 + x_3, \quad -x_6' = \epsilon^2 x_4 + \epsilon^2 x_5 + x_6,$$

where, as before $\epsilon = \exp(2\pi i/3)$. These equations, which we shall in future denote by $(x') = \chi(x)$, lead to

$$x_1' + x_2' + \ldots + x_6' = (1+2\epsilon)(x_1 + x_2 + \ldots + x_6) = 0,$$

and, if $f(x_1, \ldots, x_6) = 0$ be the equation of the primal, lead to $f(x_1', \ldots, x_6') = 9f(x_1, \ldots, x_6) = 0$, as may be verified without difficulty. The primes of the pentahedron $\{AB\}$ in the new coordinates, say

$$[AB_0]' = 0, \; [A_0 B]' = 0, \; [14]' = 0, \; [25]' = 0, \; [36]' = 0,$$

that is

$$x_1' + x_4' + \epsilon(x_3' + x_6') + \epsilon^2(x_2' + x_5') = 0, \; \ldots, \; x_1' - x_4' = 0, \; \ldots,$$

are easily seen to be given by

$$[AB_0]' = (1-\epsilon)[CA_0], \quad [A_0 B]' = (1-\epsilon)[CB_0], \quad [14]' = [CD_0],$$
$$[25]' = \epsilon[CE_0], \quad [36]' = \epsilon[CF_0].$$

Thus the primes of $\{AB\}$ are changed into the primes of $\{C_0\}$, and consequently the angular points of $\{AB\}$ into the angular points of $\{C_0\}$. For instance, the point $(C_0 D)$, or $(14.56.23)$, with coordinates $(x_1, \ldots, x_6) = (1, \epsilon^2, \epsilon^2, 1, \epsilon, \epsilon)$, gives rise to

$$x_1' = -x_4' = 3, \quad x_2' = x_3' = x_5' = x_6' = 0,$$

that is $(14)'$.

This is sufficient for our purpose. In fact the similarity of the twenty-seven pentahedra, and of the forty-five nodes, will be continually in evidence in what follows.

(4) **The Jacobian planes of the primal.** If $i$, $j$, $k$ be any three of the numbers 1, 2, ..., 6, the equations $x_i/1 = x_j/\epsilon = x_k/\epsilon^2$ represent a plane; and this plane lies entirely on the primal. For, these equations lead to $x_i+x_j+x_k = 0$, $x_jx_k+x_kx_i+x_ix_j = 0$, and, if $l$, $m$, $n$ be the three numbers of 1, 2, ..., 6 other than $i$, $j$, $k$, we also have $x_l+x_m+x_n = 0$; so that the equations identically satisfy the equation of the primal. There are twenty sets $i$, $j$, $k$; and the plane $x_i/1 = x_j/\epsilon^2 = x_k/\epsilon$ equally lies on the primal. *There are thus forty planes*, with equations of these forms, which lie on the primal.

It is found on examination that every one of these forty planes contains nine of the forty-five nodes of the primal, these nodes being arranged like the inflexions of a plane cubic curve, so that they lie in threes upon twelve lines, of which four pass through every one of the nine nodes, the joining line of any two of the nodes containing a third node, and the twelve lines form four triangles such that the nine nodes lie in threes upon the three sides of any one of these triangles. As this configuration of nine points was studied by Jacobi, we shall call any one of the forty planes a *Jacobian plane*, to facilitate reference to them.

Consider, for instance, the plane $x_1/1 = x_2/\epsilon = x_3/\epsilon^2$. This evidently contains the collinear points $P$, $Q$, $R$, respectively (56), (64), (45), for every one of which $x_1 = x_2 = x_3 = 0$. Also it contains the points

$$(B_0C), \text{ or } (15.26.34); \quad (C_0A), \text{ or } (16.24.35);$$
$$(A_0B), \text{ or } (14.25.36),$$

with respective coordinates

$$(1, \epsilon, \epsilon^2, \epsilon^2, 1, \epsilon); \quad (1, \epsilon, \epsilon^2, \epsilon, \epsilon^2, 1); \quad (1, \epsilon, \epsilon^2, 1, \epsilon, \epsilon^2),$$

which we denote by $L$, $M$, $N$. These are connected by

$$L+\epsilon^2M+\epsilon N = 0,$$

so that they are collinear.

It also contains the points $L'$, $M'$, $N'$, given respectively by

$$(EF_0),\ \text{or}\ (14.26.35);\quad (FD_0),\ \text{or}\ (16.25.34);$$
$$(DE_0),\ \text{or}\ (15.24.36),$$

with coordinates

$$(1, \epsilon, \epsilon^2, 1, \epsilon^2, \epsilon);\quad (1, \epsilon, \epsilon^2, \epsilon^2, \epsilon, 1);\quad (1, \epsilon, \epsilon^2, \epsilon, 1, \epsilon^2),$$

which equally lie on a line, since $L' + \epsilon^2 M' + \epsilon N' = 0$. And the three lines joining any one of $P$, $Q$, $R$ to the three points $L$, $M$, $N$, each contain one of the points $L'$, $M'$, $N'$. The twelve lines each containing three of the points are, in fact,

| | | | |
|---|---|---|---|
| $PQR$, | $PMN'$, | $PNL'$, | $PLM'$, |
| $LMN$, | $QNM'$, | $QLN'$, | $QML'$, |
| $L'M'N'$, | $RLL'$, | $RMM'$, | $RNN'$, |

of which the three lines in any one of the four columns contain all the nine nodes, and four of the lines pass through any one of the nodes. The existence of the nine lines containing one of the points $P$, $Q$, $R$, one of the points $L$, $M$, $N$, and one of the points $L'$, $M'$, $N'$, is at once seen if we notice such facts as that, for two points $(lp.mq.nr)$, $(lp.mr.nq)$, of which the symbols are obtained from one another by transposition of the numbers $q$, $r$, we have the identity

$$(lp.mq.nr) - (lp.mr.nq) = (\epsilon - \epsilon^2)(qr).$$

Moreover, eight such Jacobian planes pass through any one of the forty-five nodes. For through the node $(lp.mq.nr)$ there pass the eight planes $x_i/1 = x_j/\epsilon = x_k/\epsilon^2$, in which $i$ may be $l$ or $p$, and $j$ may be $m$ or $q$, and $k$ may be $n$ or $r$. While, through (for instance) the node (56), there pass the eight planes $x_i/1 = x_j/\epsilon = x_k/\epsilon^2$, $x_i/1 = x_j/\epsilon^2 = x_k/\epsilon$, in which $i$, $j$, $k$ are any three of the four numbers 1, 2, 3, 4.

If, for a moment, we use non-homogeneous coordinates, $X = 0$, $Y = 0$, $Z = 0$, $T = 0$ being four primes which pass through a chosen node, the equation of the primal will be of the form $U_2 + U_3 + U_4 = 0$, where $U_i$ is a homogeneous polynomial of order $i$ in $X$, $Y$, $Z$, $T$. And, if a plane containing this node lie entirely on the primal, any line in this plane, through this node, will lie entirely on the primal, and so will lie on every one of the cones $U_2 = 0$, $U_3 = 0$, $U_4 = 0$; the plane therefore lies on each of these cones. In particular, then, the eight Jacobian planes through the node lie on the quadric cone $U_2 = 0$, and are thus identified as the intersection of this cone with the primal. This cone we call the *asymptotic cone* at the node. A quadric cone such as $U_2 = 0$, in space of four dimensions, contains two systems of planes, each consisting of $\infty^1$ planes, with the property that two planes of the same system have in common only the point-vertex of the cone, while two planes of different systems meet in a line passing through this vertex; in fact the cone meets a threefold space in a quadric surface whose generating lines are projected from the vertex by the planes of the cone. It can at once be verified that the eight Jacobian planes through any node of the primal consist of four planes of the asymptotic cone of one system, together with four planes of the other system; so that there are sixteen lines through the node, lying on the primal, through each of which two Jacobian planes pass. For instance, consider the Jacobian planes through the node (14.25.36): the planes $x_1/1 = x_2/\epsilon = x_3/\epsilon^2$ and $x_4/1 = x_5/\epsilon = x_3/\epsilon^2$ meet only in the node, whose coordinates are $(1, \epsilon, \epsilon^2, 1, \epsilon, \epsilon^2)$, but the planes

$$x_1/1 = x_2/\epsilon = x_3/\epsilon^2 \quad \text{and} \quad x_4/1 = x_2/\epsilon = x_3/\epsilon^2$$

meet in the line $x_1/1 = x_4/1 = x_2/\epsilon = x_3/\epsilon^2$. Or again, considering the planes through the node (56), the planes $x_1/1 = x_2/\epsilon = x_3/\epsilon^2$ and $x_4/1 = x_2/\epsilon^2 = x_3/\epsilon$ meet only in the node, for which $x_1 = x_2 = x_3 = x_4 = 0$, while the planes $x_1/1 = x_2/\epsilon = x_3/\epsilon^2$ and $x_4/1 = x_2/\epsilon = x_3/\epsilon^2$ meet in the line $x_1/1 = x_4/1 = x_2/\epsilon = x_3/\epsilon^2$. The same fact will appear later from another form of the equation of the primal. Also it will appear immediately that the four lines in which a Jacobian plane, $\alpha$, through a node, is met by the four planes of the opposite system of the asymptotic cone at that node, are the lines, in the plane $\alpha$, through the node, the vertex of the cone, which each contain two other nodes of the plane $\alpha$.

The Jacobian planes considered have each twelve lines containing three nodes apiece. More generally, it is true that, if an *arbitrary* plane be drawn through any line which contains three nodes of the primal, its residual intersection with the primal is a cubic curve having an inflexion at each of the three nodes. For instance, consider the three collinear nodes (56), (64), (45), at all of which $x_1 = x_2 = x_3 = 0$. Let $a$, $b$, $c$ be arbitrary numbers and $p_1 = a+b+c$, $p_2 = bc+ca+ab$, $p_3 = abc$. The equations of the general plane containing these three nodes are $x_1/a = x_2/b = x_3/c$, and for the intersection with the primal each of these fractions is equal to $-(x_4+x_5+x_6)/p_1$. Thus, from the equation of the primal, we find that the residual intersection of the plane lies on the cubic curve obtainable from

$$p_1^3 x_4 x_5 x_6 + p_3(x_4+x_5+x_6)^3 - p_1 p_2 (x_4+x_5+x_6)(x_5x_6+x_6x_4+x_4x_5) = 0;$$

interpreting $x_4$, $x_5$, $x_6$ as coordinates in a plane, this curve has an inflexion at each of the three points $(0, 1, -1)$, $(-1, 0, 1)$, $(1, -1, 0)$, the inflexional tangent at the first of these having the equation $p_1^2 x_4 - p_2(x_4+x_5+x_6) = 0$; and so on.

(5) **The ϰ-lines of the primal.** It may easily be verified that through any line containing three nodes in a Jacobian plane there passes also another Jacobian plane. For instance, recurring to the enumeration of the nodes in the plane $x_1/1 = x_2/\epsilon = x_3/\epsilon^2$ given in the preceding §(4), the nodes $P$, $Q$, $R$ also lie on the Jacobian plane $x_1/1 = x_2/\epsilon^2 = x_3/\epsilon$, the nodes $L$, $M$, $N$ also lie on the Jacobian plane $x_4/1 = x_5/\epsilon = x_6/\epsilon^2$, and the nodes $L'$, $M'$, $N'$ on the plane $x_4/1 = x_5/\epsilon^2 = x_6/\epsilon$. Likewise the nodes $P$, $M$, $N'$, or, respectively, (56), $(1, \epsilon, \epsilon^2, \epsilon, \epsilon^2, 1)$, $(1, \epsilon, \epsilon^2, \epsilon, 1, \epsilon^2)$, lie on the plane $x_1/1 = x_3/\epsilon^2 = x_4/\epsilon$; the nodes $P$, $N$, $L'$ or, respectively, (56), $(1, \epsilon, \epsilon^2, 1, \epsilon, \epsilon^2)$, $(1, \epsilon, \epsilon^2, 1, \epsilon^2, \epsilon)$, lie on the plane $x_2/\epsilon = x_3/\epsilon^2 = x_4/1$; and the nodes $P$, $L$, $M'$, or, respectively, (56), $(1, \epsilon, \epsilon^2, \epsilon^2, 1, \epsilon)$, $(1, \epsilon, \epsilon^2, \epsilon^2, \epsilon, 1)$, lie on the plane $x_1/1 = x_2/\epsilon = x_4/\epsilon^2$; and so on—and, in virtue of the symmetry of the equation of the primal, this