I.

INTRODUCTION.

1. The notions of the 'order of greatness' or 'order of smallness' of a function $f(n)$ of a positive integral variable n, when n is 'large,' or of a function $f(x)$ of a continuous variable x, when x is 'large' or 'small' or 'nearly equal to a,' are of the greatest importance even in the most elementary stages of mathematical analysis*. The student soon learns that as x tends to infinity ($x \rightarrow \infty$) then also $x^2 \rightarrow \infty$, and moreover that x^2 tends to infinity *more rapidly than* x, *i.e.* that the ratio x^2/x tends to infinity as well; and that x^3 tends to infinity more rapidly than x^2, and so on indefinitely: and it is not long before he begins to appreciate the idea of a 'scale of infinity' (x^n) formed by the functions $x, x^2, x^3, \ldots, x^n, \ldots$. This scale he may supplement and to some extent complete by the interpolation of fractional powers of x, and, when he is familiar with the elements of the theory of the logarithmic and exponential functions, of irrational powers: and so he obtains a scale (x^α), where α is any positive number, formed by all possible positive powers of x. He then learns that there are functions whose rates of increase cannot be measured by any of the functions of this scale: that $\log x$, for example, tends to infinity more slowly, and e^x more rapidly, than *any* power of x; and that $x/(\log x)$ tends to infinity more slowly than x, but more rapidly than any power of x less than the first.

As we proceed further in analysis, and come into contact with its most modern developments, such as the theory of Fourier's series, the theory of integral functions, or the theory of singular points of analytic functions, the importance of these ideas becomes greater and

* See, for instance, my *Course of pure mathematics*, pp. 168 *et seq.*, 183 *et seq.*, 344 *et seq.*, 350.

greater. It is the systematic study of them, the investigation of general theorems concerning them and ready methods of handling them, that is the subject of Paul du Bois-Reymond's *Infinitärcalcül* or 'calculus of infinities.'

2. The notion of the 'order' or the 'rate of increase' of a function is essentially a relative one. If we wish to say that 'the rate of increase of $f(x)$ is so and so' all we can say is that it is greater than, equal to, or less than that of some other function $\phi(x)$.

Let us suppose that f and ϕ are two functions of the continuous variable x, defined for all values of x greater than a given value x_0. Let us suppose further that f and ϕ are positive, continuous, and steadily increasing functions which tend to infinity with x; and let us consider the ratio f/ϕ. We must distinguish four cases:

(i) If $f/\phi \rightarrow \infty$ with x, we shall say that the rate of increase, or simply the *increase*, of f is greater than that of ϕ, and shall write

$$f \succ \phi.$$

(ii) If $f/\phi \rightarrow 0$, we shall say that the increase of f is less than that of ϕ, and write

$$f \prec \phi.$$

(iii) If f/ϕ remains, for all values of x however large, between two fixed positive numbers δ, Δ, so that $0 < \delta < f/\phi < \Delta$, we shall say that the increase of f is equal to that of ϕ, and write

$$f \asymp \phi.$$

It may happen, in this case, that f/ϕ actually tends to a definite limit. If this is so, we shall write

$$f \overset{\asymp}{=} \phi.$$

Finally, if this limit is *unity*, we shall write

$$f \sim \phi.$$

When we can compare the increase of f with that of some standard function ϕ by means of a relation of the type $f \asymp \phi$, we shall say that ϕ *measures*, or simply *is*, the increase of f. Thus we shall say that the increase of $2x^2 + x + 3$ is x^2.

It usually happens in applications that f/ϕ is monotonic (*i.e.* steadily increasing or steadily decreasing) as well as f and ϕ themselves. It is clear that in this case f/ϕ must tend to infinity, or zero, or to a positive limit: so that one of the three cases indicated above

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must occur, and we must have $f \succ \phi$ or $f \prec \phi$ or $f \,\underline{\asymp}\, \phi$ (not merely $f \asymp \phi$). We shall see in a moment that this is not true in general.

(iv) It may happen that f/ϕ neither tends to infinity nor to zero, nor remains between fixed positive limits.

Suppose, for example, that ϕ_1, ϕ_2 are two continuous and increasing functions such that $\phi_1 \succ \phi_2$. A glance at the figure (Fig. 1) will probably show with sufficient clearness how we can construct, by means of a 'staircase' of straight or curved lines, running backwards and forwards between the graphs of ϕ_1 and ϕ_2, the graph of a steadily increasing function f such that $f=\phi_1$ for $x=x_1, x_3, \ldots$ and $f=\phi_2$ for $x=x_2, x_4, \ldots$. Then $f/\phi_1=1$ for

$$x=x_1,\ x_3,\ \ldots,$$

but assumes for $x=x_2, x_4, \ldots$ values which decrease beyond all limit; while $f/\phi_2=1$ for $x=x_2, x_4, \ldots$, but assumes for $x=x_1, x_3, \ldots$ values which increase beyond all limit; and f/ϕ, where ϕ is a function such that $\phi_1 \succ \phi \succ \phi_2$, as *e.g.* $\phi=\sqrt{(\phi_1\phi_2)}$, assumes both values which increase beyond all limit and values which decrease beyond all limit.

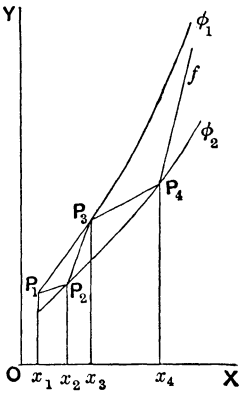

Fig. 1.

Later on (v. § 3) we shall meet with cases of this kind in which the functions are defined by explicit analytical formulae.

3. If a positive constant δ can be found such that $f > \delta\phi$ for all sufficiently large values of x, we shall write

$$f \succcurlyeq \phi;$$

and if a positive constant Δ can be found such that $f < \Delta\phi$ for all sufficiently large values of x, we shall write

$$f \preccurlyeq \phi.$$

If $f \succcurlyeq \phi$ and $f \preccurlyeq \phi$, then $f \asymp \phi$.

It is however important to observe (i) that $f \succcurlyeq \phi$ is not logically equivalent to the negation of $f \prec \phi$* and (ii) that it is not logically equivalent to the alternative '$f \succ \phi$ *or* $f \asymp \phi$.' Thus, in the example discussed at the end of § 2, $\phi_1 \succcurlyeq f \succcurlyeq \phi_2$, but no one of the relations $\phi_1 \succ f$, etc. holds. If however we know that one of the relations $f \succ \phi$, $f \asymp \phi$, $f \prec \phi$ *must* hold, then these various assertions *are* logically equivalent.

* The relations $f \succcurlyeq \phi$, $f \prec \phi$ are mutually exclusive but not exhaustive: $f \succcurlyeq \phi$ implies the negation of $f \prec \phi$, but the converse is not true.

The reader will be able to prove without difficulty that the symbols $\succ$, $\asymp$, $\prec$ satisfy the following theorems.

If $f \succ \phi$, $\phi \succcurlyeq \psi$, then $f \succ \psi$.

If $f \succcurlyeq \phi$, $\phi \succ \psi$, then $f \succ \psi$.

If $f \succcurlyeq \phi$, $\phi \succcurlyeq \psi$, then $f \succcurlyeq \psi$.

If $f \asymp \phi$, $\phi \asymp \psi$, then $f \asymp \psi$.

If $f \succcurlyeq \phi$, then $f + \phi \asymp f$.

If $f \succ \phi$, then $f - \phi \asymp f$.

If $f \succ \phi$, $f_1 \succ \phi_1$, then $f + f_1 \succ \phi + \phi_1$.

If $f \succ \phi$, $f_1 \asymp \phi_1$, then $f + f_1 \succcurlyeq \phi + \phi_1$.

If $f \asymp \phi$, $f_1 \asymp \phi_1$, then $f + f_1 \asymp \phi + \phi_1$.

If $f \succ \phi$, $f_1 \succcurlyeq \phi_1$, then $ff_1 \succ \phi\phi_1$.

If $f \asymp \phi$, $f_1 \asymp \phi_1$, then $ff_1 \asymp \phi\phi_1$.

Many other obvious results of the same character might be stated, but these seem the most important. The reader will find it instructive to state for himself a series of similar theorems involving also the symbols $\underline{\asymp}$ and $\sim$.

4. So far we have supposed that the functions considered all tend to infinity with x. There is nothing to prevent us from including also the case in which f or ϕ tends steadily to zero, or to a limit other than zero. Thus we may write $x \succ 1$, or $x \succ 1/x$, or $1/x \succ 1/x^2$. Bearing this in mind the reader should frame a series of theorems similar to those of § 3 but having reference to *quotients* instead of to sums or products.

It is also convenient to extend our definitions so as to apply to *negative* functions which tend steadily to $-\infty$ or to 0 or to some other limit. In such cases we make no distinction, when using the symbols $\succ$, $\prec$, $\asymp$, $\underline{\asymp}$, between the function and its modulus: thus we write $-x \prec -x^2$ or $-1/x \prec 1$, meaning thereby exactly the same as by $x \prec x^2$ or $1/x \prec 1$. But $f \sim \phi$ is of course to be interpreted as a statement about the actual functions and not about their moduli.

It will be well to state at this point, once for all, that all functions referred to in this tract, from here onwards, are to be understood, unless the contrary is expressly stated or obviously implied, to be positive, continuous, and monotonic, increasing of course if they tend to ∞, and decreasing if they tend to 0. But it is sometimes convenient to use our symbols even when this is not true of all the functions concerned; to write, for example,

$$1 + \sin x \prec x, \quad x^2 \succ x \sin x,$$

meaning by the first formula simply that $|1 + \sin x|/x \to 0$. This kind of use may clearly be extended even to complex functions (*e.g.* $e^{ix} \prec x$).

Again, we have so far confined our attention to functions of a continuous variable x which tends to $+\infty$. This case includes that which is perhaps even more important in applications, that of functions of the positive integral variable n: we have only to disregard values of x other than integral values. Thus $n! \succ n^2$, $-1/n \prec n$.

Finally, by putting $x = -y$, $x = 1/y$, or $x = 1/(y-a)$, we are led to consider functions of a continuous variable y which tends to $-\infty$ or 0 or a: the reader will find no difficulty in extending the considerations which precede to cases such as these.

In what follows we shall generally state and prove our theorems only for the case with which we started, that of indefinitely increasing functions of an indefinitely increasing continuous variable, and shall leave to the reader the task of formulating the corresponding theorems for the other cases. We shall in fact always adopt this course, except on the rare occasions when there is some essential difference between different cases.

5. There are some other symbols which we shall sometimes find it convenient to use in special senses.

By

$$O(\phi)$$

we shall denote a function f, otherwise unspecified, but such that

$$|f| < K\phi,$$

where K is a positive constant, and ϕ a positive function of x: this notation is due to Landau. Thus

$$x + 1 = O(x), \quad x = O(x^2), \quad \sin x = O(1).$$

We shall follow Borel in using the same letter K in a whole series of inequalities to denote a positive constant, not necessarily the same in all inequalities where it occurs. Thus

$$\sin x < K, \quad 2x + 1 < Kx, \quad x^m < Ke^x.$$

If we use K thus in any finite number of inequalities which (like the first two above) do not involve any variables other than x, or whatever other variable we are primarily considering, then all the values of K lie between certain absolutely fixed limits K_1 and K_2 (thus K_1 might be 10^{-10} and K_2 be 10^{10}). In this case all the K's satisfy $0 < K_1 < K < K_2$, and every relation $f < K\phi$ might be replaced by $f < K_2\phi$, and every relation $f > K\phi$ by $f > K_1\phi$. But we shall also have occasion to use K in equalities which (like the third above) involve a parameter (here m). In this case K, though independent of x, is a function of m. Suppose that α, β, ... are all the parameters which occur in this way in this tract. Then if we give any special system of values to α, β, ..., we can determine K_1, K_2 as above. Thus all our K's satisfy

$$0 < K_1(\alpha, \beta, \ldots) < K < K_2(\alpha, \beta, \ldots),$$

where K_1, K_2 are positive functions of α, β, ... defined for any permissible set of values of those parameters. But K_1 has zero for its lower limit; by choosing α, β, ... appropriately we can make K_1 as small as we please—and, of course, K_2 as large as we please*.

It is clear that the three assertions

$$f = O(\phi), \quad |f| < K\phi, \quad f \preccurlyeq \phi$$

are precisely equivalent to one another.

When a function f possesses any property for all values of x greater than some definite value (this value of course depending on the nature of the particular property) we shall say that f possesses the property for $x > x_0$. Thus

$$x > 100 \ \ (x > x_0), \quad e^x > 100\,x^2 \ \ (x > x_0).$$

We shall use δ to denote an arbitrarily small but fixed positive number, and Δ to denote an arbitrarily great but likewise fixed positive number. Thus

$$f < \delta\phi \quad (x > x_0)$$

means 'however small δ, we can find x_0 so that $f < \delta\phi$ for $x > x_0$,' *i.e.* means the same as $f \prec \phi$; and $\phi > \Delta f$ $(x > x_0)$ means the same: and

$$(\log x)^\Delta \prec x^\delta$$

means 'any power of $\log x$, however great, tends to infinity more slowly than any positive power of x, however small.'

Finally, we denote by ϵ a function (of a variable or variables indicated by the context or by a suffix) whose limit is zero when the variable or variables are made to tend to infinity or to their limits in the way we happen to be considering. Thus

$$f = \phi(1 + \epsilon), \ f \sim \phi$$

are equivalent to one another.

* I am indebted to Mr Littlewood for the substance of these remarks.

In order to become familiar with the use of the symbols defined in the preceding sections the reader is advised to verify the following relations; in them $P_m(x)$, $Q_n(x)$ denote polynomials whose degrees are m and n and whose leading coefficients are positive:

$$P_m(x) \succ Q_n(x)\ (m > n), \quad P_m(x) \asymp Q_n(x)\ (m = n),$$

$$P_m(x) \asymp x^m, \quad P_m(x)/Q_n(x) \asymp x^{m-n},$$

$$\sqrt{(ax^2+2bx+c)} \asymp x\ (a > 0), \quad \sqrt{(x+a)} \sim \sqrt{x},$$

$$\sqrt{(x+a)} - \sqrt{x} \sim a/2\sqrt{x}, \quad \sqrt{(x+a)} - \sqrt{(x)} = O(1/\sqrt{x}),$$

$$e^x \succ x^\Delta, \quad e^{x^2} \succ e^{\Delta x}, \quad e^{e^x} \succ e^{x^\Delta},$$

$$\log x \prec x^\delta, \quad \log P_m(x) \asymp \log Q_n(x), \quad \log\log P_m(x) \sim \log\log Q_n(x),$$

$$x + a\sin x \sim x, \quad x(a+\sin x) \asymp x\ (a > 1),$$

$$e^{a+\sin x} \asymp 1, \quad \cosh x \sim \sinh x \asymp e^x,$$

$$x^m = O(e^{\delta x}), \quad (\log x)/x = O(x^{\delta-1}),$$

$$1+\frac{1}{2}+\ldots+\frac{1}{n} \succ 1, \quad 1+\frac{1}{2^2}+\ldots+\frac{1}{n^2} \asymp 1,$$

$$1+\frac{1}{2}+\ldots+\frac{1}{n} \sim \log n, \quad 1+\frac{1}{2}+\ldots+\frac{1}{n} - \log n \asymp 1,$$

$$n! \prec n^n, \quad n! \succ e^{\Delta n}, \quad n! = n^{n^{1+\epsilon}} = n^{n(1+\epsilon)},$$

$$n! \sim n^{n+\frac{1}{2}} e^{-n} \sqrt{(2\pi)}, \quad n!\,(e/n)^n = (1+\epsilon)\sqrt{(2\pi n)},$$

$$\int_1^x \frac{dt}{t} \succ 1, \quad \int_1^x \frac{dt}{t} \sim \log x, \quad \int_2^x \frac{dt}{\log t} \sim \frac{x}{\log x}.$$

II.

SCALES OF INFINITY IN GENERAL.

1. If we start from a function ϕ, such that $\phi \succ 1$, we can, in a variety of ways, form a series of functions

$$\phi_1 = \phi, \quad \phi_2, \quad \phi_3, \ \ldots, \ \phi_n, \ \ldots$$

such that the increase of each function is greater than that of its predecessor. Such a sequence of functions we shall denote for shortness by (ϕ_n).

One obvious method is to take $\phi_n = \phi^n$. Another is as follows: If $\phi \succ x$, it is clear that

$$\phi\{\phi(x)\}/\phi(x) \to \infty,$$

and so $\phi_2(x) = \phi\phi(x) \succ \phi(x)$; similarly $\phi_3(x) = \phi\phi_2(x) \succ \phi_2(x)$, and so on*.

Thus the first method, with $\phi = x$, gives the scale $x, x^2, x^3, \ldots$ or (x^n); the second, with $\phi = x^2$, gives the scale $x^2, x^4, x^8, \ldots$ or (x^{2^n}).

These scales are *enumerable* scales, formed by a simple progression of functions. We can also, of course, by replacing the integral parameter n by a continuous parameter a, define scales containing a non-enumerable multiplicity of functions: the simplest is (x^a), where a is any positive number. But such scales fill a subordinate *rôle* in the theory.

It is obvious that we can always insert a new term (and therefore, of course, any number of new terms) in a scale at the beginning or between any two terms: thus $\sqrt{\phi}$ (or ϕ^a, where a is any positive number less than unity) has an increase less than that of any term of the scale, and $\sqrt{(\phi_n \phi_{n+1})}$ or $\phi_n^a \phi_{n+1}^{1-a}$ has an increase intermediate between those of ϕ_n and ϕ_{n+1}. A less obvious and far more important theorem is the following

Theorem of Paul du Bois-Reymond. *Given any ascending scale of increasing functions ϕ_n, i.e. a series of functions such that $\phi_1 \prec \phi_2 \prec \phi_3 \prec \ldots$, we can always find a function f which increases more rapidly than any function of the scale, i.e. which satisfies the relation $\phi_n \prec f$ for all values of n.*

In view of the fundamental importance of this theorem we shall give two entirely different proofs.

2. (i) We know that $\phi_{n+1} \succ \phi_n$ for all values of n, but this, of course, does not necessarily imply that $\phi_{n+1} \geqslant \phi_n$ for all values of x and n in question†. We can, however, construct a new scale of functions ψ_n such that

(*a*) ψ_n is identical with ϕ_n for all values of x from a certain value x_n onwards (x_n, of course, depending upon n);

(*b*) $\psi_{n+1} \geqslant \psi_n$ for all values of x and n.

For suppose that we have constructed such a scale up to its nth term ψ_n. Then it is easy to see how to construct ψ_{n+1}. Since $\phi_{n+1} \succ \phi_n$, $\phi_n \sim \psi_n$, it follows that $\phi_{n+1} \succ \psi_n$, and so $\phi_{n+1} > \psi_n$ from a certain value of x (say x_{n+1}) onwards. For $x \geqslant x_{n+1}$ we take $\psi_{n+1} = \phi_{n+1}$. For $x < x_{n+1}$ we give ψ_{n+1} a value equal to the greater of the values of

* For some results as to the increase of such iterated functions see VII. § 2 (vi).

† $\phi_{n+1} \succ \phi_n$ implies $\phi_{n+1} > \phi_n$ for sufficiently large values of x, say for $x > x_n$. But x_n may tend to ∞ with n. Thus if $\phi_n = x^n/n!$ we have $x_n = n+1$.

ϕ_{n+1}, ψ_n. Then it is obvious that ψ_{n+1} satisfies the conditions (*a*) and (*b*).

Now let $$f(n) = \psi_n(n).$$

From $f(n)$ we can deduce a continuous and increasing function $f(x)$, such that

$$\psi_n(x) < f(x) < \psi_{n+1}(x)$$

for $n < x < n + 1$, by joining the points $(n, \psi_n(n))$ by straight lines or suitably chosen arcs of curves.

It is perhaps worth while to call attention explicitly to a small point that has sometimes been overlooked (see, *e.g.*, Borel, *Leçons sur la théorie des fonctions*, p. 114; *Leçons sur les séries à termes positifs*, p. 26). It is not always the case that the use of straight lines will ensure

$$f(x) > \psi_n(x)$$

for $x > n$ (see, for example, Fig. 2, where the dotted line represents an appropriate arc).

Then $$f/\psi_n > \psi_{n+1}/\psi_n$$

for $x > n + 1$, and so $f \succ \psi_n$; therefore $f \succ \phi_n$, and the theorem is proved.

The proof which precedes may be made more general by taking $f(n) = \psi_{\lambda_n}(n)$, where λ_n is an integer depending upon n and tending steadily to infinity with n.

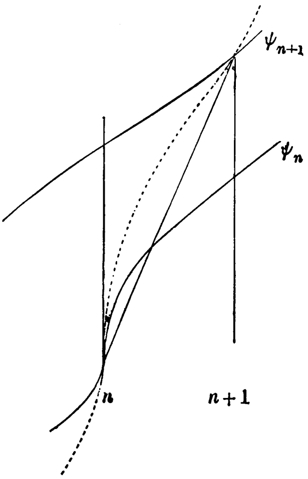

FIG. 2.

(ii) The second proof of Du Bois-Reymond's Theorem proceeds on entirely different lines. We can always choose positive coefficients a_n so that

$$f(x) = \sum_1^\infty a_n \psi_n(x)$$

is convergent for all values of x. This will certainly be the case, for instance, if

$$1/a_n = \psi_1(1)\psi_2(2) \dots \psi_n(n).$$

For then, if ν is any integer greater than x, $\psi_n(x) < \psi_n(n)$ for $n \geqq \nu$, and the series will certainly be convergent if

$$\sum_\nu^\infty \frac{1}{\psi_1(1)\psi_2(2) \dots \psi_{n-1}(n-1)}$$

is convergent, as is obviously the case.

Also $$f(x)/\psi_n(x) > a_{n+1}\psi_{n+1}(x)/\psi_n(x) \to \infty,$$
so that $f \succ \phi_n$ for all values of n.

3. Suppose, *e.g.*, that $\phi_n = x^n$. If we restrict ourselves to values of x greater than 1, we may take $\psi_n = \phi_n = x^n$. The first method of construction would naturally lead to

$$f = n^n = e^{n \log n},$$

or $f = (\lambda_n)^n$, where λ_n is defined as at the end of § 2 (i), and each of these functions has an increase greater than that of any power of n. The second method gives

$$f(x) = \sum_1^\infty \frac{x^n}{1^1 2^2 3^3 \dots n^n}.$$

It is known* that when x is large the order of magnitude of this function is roughly the same as that of

$$e^{\frac{1}{2}(\log x)^2/\log\log x}.$$

As a matter of fact it is by no means necessary, in general, in order to ensure the convergence of the series by which $f(x)$ is defined, to suppose that a_n decreases so rapidly. It is very generally sufficient to suppose $1/a_n = \phi_n(n)$: this is always the case, for example, if $\phi_n(x) = \{\phi(x)\}^n$, as the series

$$\Sigma \left\{\frac{\phi(x)}{\phi(n)}\right\}^n$$

is always convergent. This choice of a_n would, when $\phi = x$, lead to

$$f(x) = \Sigma \left(\frac{x}{n}\right)^n \sim \sqrt{\left(\frac{2\pi x}{e}\right)}\, e^{x/e}\,\dagger.$$

But the simplest choice here is $1/a_n = n!$, when

$$f(x) = \Sigma \frac{x^n}{n!} = e^x - 1;$$

it is naturally convenient to disregard the irrelevant term -1.

4. We can always suppose, if we please, that $f(x)$ is defined by a power series $\Sigma a_n x^n$ convergent for all values of x, in virtue of a theorem of Poincaré's ‡ which is of sufficient intrinsic interest to deserve a formal statement and proof.

Given any continuous increasing function $\phi(x)$, we can always find an integral function $f(x)$ (i.e. a function $f(x)$ defined by a power series $\Sigma a_n x^n$ convergent for all values of x) such that $f(x) \succ \phi(x)$.

The following simple proof is due to Borel §.

Let $\Phi(x)$ be any function (such as the square of ϕ) such that $\Phi \succ \phi$. Take

* *Messenger of Mathematics*, vol. 34, p. 101.

† Lindelöf, *Acta Societatis Fennicae*, t. 31, p. 41; Le Roy, *Bulletin des Sciences Mathématiques*, t. 24, p. 245.

‡ *American Journal of Mathematics*, vol. 14, p. 214.

§ *Leçons sur les séries à termes positifs*, p. 27.