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W. H. Young

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THE FUNDAMENTAL THEOREMS
OF THE
DIFFERENTIAL CALCULUS

by

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PREFACE

IN this Tract rigidity of proof and novelty of treatment have been aimed at rather than simplicity of presentation, though this has never been lightly sacrificed. The Differential Calculus is concerned with those continuous functions that possess differential coefficients and with these differential coefficients themselves. As a differential coefficient is not necessarily a continuous function, the subject merges naturally into the wider one of the Theory of Functions of one or more Real Variables, and cannot, therefore, be completely mastered without some knowledge of the Theory of Sets of Points. No more knowledge of the language or concepts of this theory will however here be required than a serious mathematical student may now be supposed to have gained before completing his Degree course, and, with this exception, the present account of the fundamental theorems of the Differential Calculus will, it is hoped, be found to be complete in itself. For the rest a brief account is given in Appendix III of the definitions and results from the Theory of Sets of Points actually employed in the Tract.

The theory of those functions that correspond to the differential coefficient at a point at which this latter does not exist, does not fall naturally within our scope. Some of the remarkable properties of these interesting functions, with other generalities, are, however, stated without proof in Appendix I.

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PREFACE

The theory of Maxima and Minima has been barely alluded to, and the complex variable has been rigorously excluded. Apart from other considerations the space at our disposal has here been decisive. For the same reason it has been impossible to give more than a few isolated examples. Appendix II consists of references to some of the existing literature, where such examples may be found.

It is perhaps well to warn the English reader in conclusion that such initial difficulties as he may feel are likely to be due, in part at least, to a lack of familiarity with the modern formulation of the concept of an irrational number.

W. H. YOUNG.

LA NONETTE DE LA FORÊT,
GENEVA, SWITZERLAND.
November, 1909.

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