CHAPTER I

THE POSITION AND SIZE OF THE IMAGE.

1. Rays and waves of light.

The existence of "shadows," which is constantly observed in everyday life, is most simply explained by the supposition that the influence to which our eyes are sensitive, and which we call *light*, travels (at any rate in air) in straight lines issuing in all directions from the "luminous" bodies with which it originates, and that it can be stopped by certain obstacles which are called *opaque*. This supposition of the *rectilinear propagation of light* is not exactly confirmed by more precise observations: light does in fact bend round the corners of opaque bodies to a certain very small extent. But the supposition is so close an approximation to the truth that it may be taken as exact without sensibly invalidating the discussion and explanation of many of the most noteworthy phenomena of light.

If an opaque screen, pierced by a small hole, be placed at some distance from a small source of light, the light transmitted through the hole will therefore travel approximately in the prolongation of the straight line joining the source to the hole. Light which is isolated in this way, so as to have approximately a common direction of propagation, is called a *pencil*: and a luminous body is to be regarded as sending out pencils of light in all directions. As there is a certain amount of vagueness in this statement, owing to the absence of any definite understanding as to what the cross-section of a pencil is to be, it is customary to make use of that principle of idealisation which is of such constant occurrence in mathematics: we introduce the term *ray* to denote a pencil whose cross-section is infinitesimally small, so that the light can be regarded as confined to a straight line: and then the above idea can be expressed by the statement that *a luminous body sends out rays of light in all directions.*

A more intimate study of the physical properties of light tends to the conviction that light consists in a disturbance of a medium which fills all space, interpenetrating material bodies: to this medium the name *aether* is given. A luminous point is then to be regarded as sending out waves of disturbance into the surrounding aether, in much the same fashion as a stone dropped into a pond sends out waves of disturbance in the water of the pond. In the latter case, we can distinguish between the *crests* of the waves, where the water is heaped up, and the *troughs*, where the surface is depressed below the normal level: these crests and troughs form a system of circles having for centre the point where the stone struck the water: we can speak of any crest or trough, or indeed any circle which has this point for centre, as a *wave-front*, meaning thereby that at all points of such a circle the water is at any instant in the same phase of disturbance. Similarly in the case of the waves emitted by a luminous point in any medium which is homogeneous (*i.e.* has the same properties at all its points) and isotropic (*i.e.* has the same properties with respect to all directions), the aether is in the same phase of disturbance at any instant at all points of a sphere having the luminous point as centre: and these surfaces of equal phase are called *wave-fronts*. It is evident that *the rays of light proceeding from the point are simply the normals to the wave-fronts.*

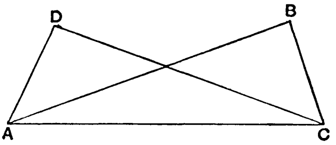
The luminous disturbances with which we are familiar in nature are generally of a very complicated character, but can be regarded as formed by the coexistence of a number of disturbances of simpler type, in which those wave-fronts which have the same phase (*e.g* the "crests") follow each other at regular intervals of distance. This distance is called the *wave-length* of the simple disturbance: and the time taken by one crest to move over one wave-length, *i.e.* to replace the crest in front of it, is called the *period*. Differences of wave-length or period affect the eye as differences of *colour*.

The wave-fronts are propagated outwards from a luminous point, in the same way as the water-waves on the pond: the velocity with which a wave-front moves along its own normal depends on the material medium (*e.g.* air or glass) in which the propagation is taking place. The ratio of the velocity of light *in vacuo* to the velocity in any given medium is called the *index of refraction* of the medium: it is proportional to the time light takes to travel 1 cm. in the medium. The refractive index depends to some extent on the colour of the light considered: we shall suppose for the present that we are dealing with light of some definite period, so that the index of refraction has a definite value for every medium considered.

2. Reflexion.

It is a familiar fact that light is to some extent thrown back or *reflected* from the surfaces of most bodies on which it is incident. In most cases the incident wave-front is so broken up by the small irregularities of surface of the reflecting body, that any regularity which it may have possessed before reflexion is destroyed: but if the reflecting body is capable of being used as a *mirror*, *i.e.* if its surface is optically smooth, reflexion has a regular character which we shall now investigate.

Let the plane of the diagram be perpendicular to the reflecting surface and the incident wave-front, and let AC, AB be the traces of the reflecting surface and the incident wave-front respectively. Let DC be the trace of the wave-front after reflexion, and let BC and AD be perpendicular to the respective wave-fronts, so that they are respectively parallel to the incident and reflected beams of light.



Then the time taken by the wave-front to travel from one position to the other is proportional to either BC (which represents the time taken by B to move to its new position C) or to AD (which represents the time taken by A to move to its new position D): we have therefore

$$BC = AD, \quad \text{or} \quad B\hat{C}A = D\hat{A}C.$$

The angle between the incident ray BC and the normal to the surface is called the *angle of incidence*: the angle between the emergent ray AD and the normal is called the *angle of reflexion*. The last equation may be expressed by the statement that *the reflected ray is in the same plane as the incident ray and the normal to the reflecting surface, and the angle of reflexion is equal to the angle of incidence.* This is the *law of reflexion.*

3. Refraction: Fermat's principle.

If a thick piece of glass or any other transparent substance be interposed in air between a luminous body and the eye, the luminous source will in general still be seen, but will appear distorted or displaced in some manner. From this it is evident that while the rays from the luminous body which strike the glass are in part reflected at the surface of the glass, they are also partly transmitted through the glass, and at the same time experience a certain amount of deflexion

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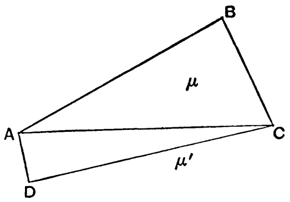
from their original course. It can easily be shewn experimentally that this deflexion, to which the name *refraction* is given, takes place at the entry of the ray into the glass, and again at its emergence from the glass: there is no change of direction of the ray during its passage through the glass, if the latter be homogeneous.

If a ray of light passes from one medium into another, the acute angle between the incident ray and the normal to the interface between the media is called the *angle of incidence*, and the acute angle between the refracted ray and the normal is called the *angle of refraction*.

Refraction is easily explained as a consequence of the difference of velocity of propagation of light in different media. Let AC be the trace of a small part of the refracting surface: let AB be the trace of the incident wave-front, so that its normal BC is parallel to the incident beam: let DC be the trace of the wave-front after refraction, and AD its normal: and let μ and μ' denote the refractive indices of the media.



Then the time taken by the wave-front to travel from one position to the other is proportional to $\mu . BC$ (which represents the time taken by B in travelling to C) or to $\mu' . AD$ (which represents the time taken by A in travelling to D). We have therefore,

$$\mu . BC = \mu' . AD, \quad \text{or} \quad \mu \sin B\hat{A}C = \mu' \sin A\hat{C}D.$$

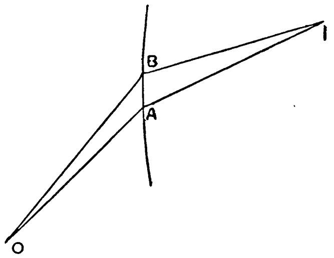
Thus the *law of refraction* is that *the sines of the angles of incidence and refraction are in the ratio* μ'/μ. This is readily seen to be equivalent to the statement that the cosines of the angles made by the incident and refracted rays with any line in the tangent-plane to the interface are in the ratio μ'/μ.

Media for which the index of refraction has comparatively large or small values are spoken of as *optically dense* or *optically light* respectively.

When the refraction takes place from a dense into a light medium, so that $\mu > \mu'$, the law of refraction gives a real value for the angle of refraction only when the angle of incidence is less than $\sin^{-1}(\mu'/\mu)$. This value of the angle of incidence is called the *critical angle*: when the angle of incidence is greater than the critical angle, refraction does not take place, all the light being reflected. This phenomenon is known as *total internal reflexion*.

The laws of reflexion and refraction can be comprehended in a single statement known as *Fermat's principle*, which may be thus stated: *The path which is actually described by a ray of light between two points is such that the time taken by light in travelling from one point to the other is stationary* (*i.e.* is a maximum or minimum) *for that path as compared with adjacent paths connecting the same terminal points*: the velocity of the light being everywhere proportional inversely to the refractive index. In the case of reflexion the condition must of course be added that the path of the ray is to meet the reflecting surface.

To shew that Fermat's principle is equivalent to the ordinary law of refraction, let OA be an incident ray in a medium of index μ, AI the refracted ray in a medium of index μ', B any point near to A on the refracting surface AB. The excess of length of OB over OA is evidently $AB \cos O\hat{B}A$, and the excess of length of AI over BI is $AB \cos B\hat{A}I$: so the difference between the times of propagation of luminous disturbance along the two paths OBI and OAI is proportional to



$$\mu \,.\, AB \cos O\hat{B}A - \mu' \,.\, AB \cos B\hat{A}I,$$

which vanishes in consequence of the law of refraction: this establishes the stationary property which is enunciated in Fermat's principle.

Fermat's principle is analytically expressed by the statement that

$$\int \mu ds$$

(where μ denotes the refractive index for the element ds of the path) has a stationary value, when the integration is taken along the actual path of a ray between two given terminals, as compared with adjacent curves connecting the same terminals.

4. Object and image.

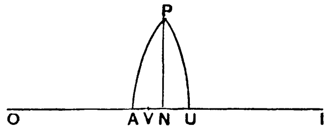
In the preceding discussion of reflexion and refraction we have considered only the direction of the tangent-plane to a wave-front at some particular point: we must now proceed to consider the curvature of the wave-front, which of course depends on its distance from the luminous point from which it is diverging. The same idea can be otherwise expressed by the statement that we have hitherto treated only single *rays*, but are now about to study *pencils*.

Consider a luminous point which is emitting waves in air; we shall call this the *object-point*. Suppose that the light, after proceeding some distance from the object-point, is incident almost perpendicularly on a convex lens (*i.e.* a piece of glass bounded by two spherical faces and thickest in the middle). The waves before incidence on the lens are convex in front, so that the part of the wave-front which strikes the centre of the lens is originally a little ahead of the parts of the wave-front which strike the rim of the lens: but as the luminous disturbance travels more slowly in the glass than in air, that part of the wave which passes through the centre of the lens, and therefore has the greatest thickness of glass to traverse, will be retarded relatively to the outer parts of the wave in passing through the lens; and it may happen that this takes place to such an extent as to make the outer portions of the wave-front ahead of the central portion when the wave emerges from the lens, so that the wave is now concave in front. This concave wave will propagate itself onwards, in the direction of its own normal at every point, and thus its radius of curvature will gradually decrease until the wave finally converges to a point. This point, to which the luminous disturbance issuing from the object-point and caught by the lens is now ingathered, is said to be a *real image* of the original object-point.

In any case the centre of curvature of the wave-fronts after emergence from the lens is said to be an *image* of the object-point, the image being called *virtual* if the luminous disturbance does not actually pass through it.

5. Image-formation by direct refraction at the spherical interface between two media.

The fundamental case of image-formation is that in which the light issuing from an object is refracted at a spherical interface between two media. Let the refractive indices of the first and second media be μ and μ' respectively, and let r be the radius of curvature of the interface, counted positively when the surface is convex to the incident light. Let O be the object-point, A the *vertex* or foot of the normal from O to the interface, P a point on the interface near A, PN perpendicular to the *axis* or central line OA. We shall consider the formation of an image by a luminous disturbance which is propagated approximately along the axis.



A spherical wave-front originating

from O would, but for its encounter with the second medium, occupy at some time a position represented by the trace PU, where U is a point on the axis such that $OU = OP$. But owing to the fact that the disturbance does not travel with the same velocity in the two media, the disturbance along the axis will have reached only to a point V, where

$$\mu' \,.\, AV = \mu \,.\, AU$$

or

$$(\mu' - \mu)\, AN - \mu \,.\, NU = \mu' \,.\, VN.$$

But by a well-known property of circles, we have

$$PN^2 = NU\,(ON + OU), \quad \text{and} \quad PN^2 = NA(2r - NA),$$

and the equation can therefore be written in the form

$$\frac{(\mu' - \mu)\,.\,PN^2}{2r - NA} - \frac{\mu\,.\,PN^2}{ON + OU} = \mu' \,.\, VN,$$

which when P approaches indefinitely near to A takes the form

$$\frac{\mu' - \mu}{r} - \frac{\mu}{OA} = \frac{2\mu' \,.\, VN}{PN^2},$$

shewing that V and P lie on a sphere of centre I, where

$$\frac{\mu' - \mu}{r} - \frac{\mu}{OA} = \frac{\mu'}{AI}.$$

This sphere evidently represents the wave-front after refraction, and *its centre* I, determined by the last equation, *is the image-point corresponding to the object-point* O. This equation shews that the range formed by any number of object-points on the line OAI is, in the language of geometry, homographic with the range formed by the corresponding image-points.

6. Image-formation by direct refraction at any number of spherical surfaces on the same axis.

We shall consider next the successive refraction of a pencil of light at any number of spherical refracting surfaces whose centres of curvature are on the same line or *axis*; the object-point will be supposed for the present to be also situated on this axis, and the pencil of light to be directed approximately along the axis.

Let x denote the abscissa of the object-point, measured (positively in the direction of propagation of the light) from any fixed origin on the axis: and let the abscissae of the successive images be $x_1, x_2, \ldots, x'$.

Then the homographic property found in § 5 shews that x_1 is given in terms of x by an equation which can be written in the general form

$$x_1 = \frac{\alpha_1 x + \beta_1}{\gamma_1 x + \delta_1},$$

where $(\alpha_1, \beta_1, \gamma_1, \delta_1)$ are constants which depend on the position and curvature of the first refracting surface and on the refractive indices of the first and second media.

Similarly the positions of the successive images are given by equations which may be written in the form

$$x_2 = \frac{\alpha_2 x_1 + \beta_2}{\gamma_2 x_1 + \delta_2}, \qquad x_3 = \frac{\alpha_3 x_2 + \beta_3}{\gamma_3 x_2 + \delta_3}, \quad \ldots\ldots .$$

Combining these so as to eliminate the intermediate images, we see that the position x' of the final image-point is determined in terms of the position x of the original object-point by an equation which can also be written in the form

$$x' = \frac{\alpha x + \beta}{\gamma x + \delta}$$

where $(\alpha, \beta, \gamma, \delta)$ are constants depending on the system of refracting surfaces, but not depending on the position of the object-point.

If γ is zero, the system is said to be a *telescopic system*: the equation which determines x' in terms x' then becomes

$$x' = \frac{\alpha}{\delta} x + \frac{\beta}{\delta},$$

which by change of origin can be written

$$x' = kx,$$

where k is a constant.

If γ is not zero (which is the more general case), we can evidently without loss of generality take γ to be unity: the equation can then be written

$$xx' + \delta x' - \alpha x - \beta = 0;$$

so if we now measure x from a point at a distance $-\delta$ from the original origin, and also measure x' from a point at a distance α from the original origin, the equation will take the form

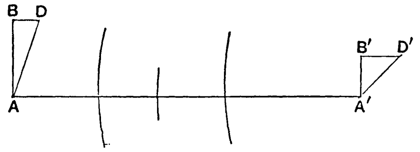
$$xx' = C,$$

where C is a constant. *This equation determines the position x' of the final image.* The origin from which x is now measured is called the *First Principal Focus* of the optical system: it is evidently the position in which the object must be placed in order that the image may be at an infinite distance, *i.e.* in order that the emergent wave-fronts may be plane. Similarly the origin from which x' is measured is called the *Second Principal Focus*: it is the position taken by the image-point when the object-point is at an infinite distance, *e.g.* a star. In the accustomed language of geometry, the Principal Foci are the "vanishing points" of the homographic ranges formed by any set of object-points and the corresponding image-points.

7. The Helmholtz-Clausius equation.

The equation $$xx' = C$$
determines the *position* x' of the image formed by a given optical system, in terms of the position x of the object: we shall next shew how to determine the *size* of the image in terms of the size and position of the object, when the latter is supposed to be no longer a point but a body of finite (though small) dimensions.

Let AB be an object, perpendicular to the axis AA' of the instrument, and let $A'B'$ be its image; we can regard AB and $A'B'$ as two



positions of a wave-front, when small quantities of the second order are neglected (the ratio of the height AB to the dimensions of the instrument being taken as a small quantity of the first order). Let AD, $A'D'$, be the corresponding two positions of another wave-front (proceeding of course from another source) slightly inclined to the first.

Then the time taken by the luminous disturbance to travel from B to B'

= ,, ,, ,, ,, ,, ,, ,, A to A'

= ,, ,, ,, ,, ,, ,, ,, D to D'.

It follows that the time taken by the light to travel the distance BD in the initial medium is equal to the time taken to travel $B'D'$ in the final medium: or
$$\mu . BD = \mu' . B'D',$$
where μ and μ' are the refractive indices of the initial and final media.

If then we denote the heights AB, $A'B'$ of the object and image by y and y' respectively, and the initial and final angles $B\hat{A}D$, $B'\hat{A}'D'$ between slightly inclined wave-fronts by α, α', respectively we have
$$\mu y \alpha = \mu' y' \alpha'.$$
This is known as *Helmholtz's equation*: it gives the *linear magnification* y'/y in terms of the *angular magnification* α'/α.

It is obvious that the above reasoning does not depend essentially on the circumstance that the optical instrument has been supposed to be symmetrical about an axis: we can therefore abandon this supposition, and state the theorem in a more general form due to Clausius*. Suppose that a small line-element l in a medium of index μ has for image a small line-element l' in a medium of index μ', and that a pencil of light which has a small angular aperture α when it issues from a point of l has an aperture α' when it converges to the corresponding image-point on l': and let ψ and ψ' be the angles made by l and l' respectively with the normals to the pencil in its plane at the two ends. Then $l \cos \psi$ will correspond to the y of Helmholtz's equation, and $l' \cos \psi'$ to y': so we obtain *Clausius' equation*

$$\mu l \alpha \cos \psi = \mu' l' \alpha' \cos \psi'.$$

8. The transformation of the object-space into the image-space.

We are now in a position to obtain formulae which completely determine the manner in which an optical instrument forms an image of a small object situated on its axis of symmetry.

The position of any point of a possible object, or any *point of the object-space* as it is generally called, will be specified by its abscissa x measured along the axis (positively in the direction of propagation of the light) from the First Principal Focus of the instrument, and its ordinate y drawn perpendicularly to the axis: and similarly the position of a point in the image-space will be specified by coordinates (x', y'), of which x' is measured from the Second Principal Focus of the instrument.

Suppose that two objects, of heights y_1, y_2 respectively, are at the points whose abscissae are x_1, x_2: let their images be of heights y_1', y_2', respectively. Then the equation of § 6 gives

$$xx' = C,$$

so we have

$$\text{Distance between images} = \frac{C}{x_2} - \frac{C}{x_1} = -\frac{C}{x_1 x_2} \times \text{Distance between objects.}$$

If therefore α denote the inclination to the axis of the ray from the axial point of the first object to the topmost point of the second object, and if α' denote the inclination of this ray to the axis after passing through the instrument, we have

$$\frac{\alpha'}{\alpha} = \frac{y_2'}{y_2} \times \frac{\text{Distance between objects}}{\text{Distance between images}}$$
$$= -\frac{y_2'}{y_2} \frac{x_1 x_2}{C}.$$

* *Ann. der Phys.* CXXI. (1864), 1.