Cambridge University Press
978-1-107-49296-7 - The Bloch–Kato Conjecture for the Riemann Zeta Function: London Mathematical Society Lecture Note Series: 418
Edited by John Coates, A. Raghuram, Anupam Saikia and R. Sujatha
Excerpt
More information

1

Special Values of the Riemann Zeta Function: Some Results and Conjectures

A. Raghuram

Abstract

These notes are based on two lectures given at the instructional workshop on the Bloch–Kato conjecture for the values of the Riemann ζ-function at odd positive integers. The workshop was held at IISER, Pune, in July 2012. The aim of these notes is to give a brief introduction to (i) Borel's results and Lichtenbaum's conjectures on the special values of the Riemann ζ-function, and (ii) Deligne's conjecture and the Tamagawa number conjecture of Bloch and Kato on the special values of motivic L-functions as applied to Tate motives.

1.1 Values of the Riemann ζ-function and K-groups of $\mathbb{Z}$

1.1.1 Definition and basic analytic properties of $\zeta(s)$

Definition of $\zeta(s)$

The Riemann zeta function is defined by the series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \sigma > 1,$$

where $s = \sigma + it$ is a complex variable with $\sigma = \Re(s)$ and $t = \Im(s)$. The series converges absolutely for $\sigma > 1$, which may be seen using the integral test by comparing the series with $\int_1^\infty 1/x^\sigma \, dx$.

Indian Institute of Science Education and Research (IISER), Pune, Maharashtra 411021, India.
e-mail: raghuram@iiserpune.ac.in

Euler product, analytic continuation and functional equation

Theorem 1.1.1 (Basic Analytic Properties) *The Riemann zeta function has the following properties:*

1. *(Euler Product) For $\sigma > 1$ we have*

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1},$$

where the product runs over all primes p.

2. *(Analytic continuation) The function $\zeta(s)$, which is defined for $\sigma > 1$, extends to a meromorphic function to all of $\mathbb{C}$ with only one pole which is located at $s = 1$ and is a simple pole with residue 1. Around $s = 1$, we have*

$$\zeta(s) = \frac{1}{s-1} + \gamma + \sum_{k=1}^{\infty} \gamma_k (s-1)^k$$

where γ is Euler's constant.

3. *(Functional equation)*

$$\pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right)\zeta(s) \;=\; \pi^{(1-s)/2}\,\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s),$$

where $\Gamma(s)$ is the usual Γ-function.

See, for example, Ivic [Iv85, Chapter 1].

Let $\zeta_\infty(s) = \pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right)$, and define the completed zeta function by

$$\Lambda(s) \;:=\; \zeta_\infty(s)\zeta(s) \;=\; \pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right)\zeta(s). \tag{1.1}$$

Then $\Lambda(s)$ has a meromorphic continuation to all of $\mathbb{C}$ with simple poles at $s = 0, 1$ and is holomorphic elsewhere. The functional equation looks like $\Lambda(s) = \Lambda(1-s)$. In analytic number theory, one also completes $\zeta(s)$ as $\Lambda^*(s) = s(1-s)\Lambda(s)$. We still have the same functional equation $\Lambda^*(s) = \Lambda^*(1-s)$, but $\Lambda^*(s)$ has the virtue of being an entire function. However, from the motivic or automorphic perspective, the completed zeta function is always taken to be $\Lambda(s)$ and not $\Lambda^*(s)$.

An easy consequence of the functional equation is:

$$\zeta(0) \;=\; -\frac{1}{2}. \tag{1.2}$$

(For the interested reader, here is a quote from Ramanujan's Notebooks: *The constant of a series has some mysterious connection with the given infinite series and it is like the centre of gravity of a body. Mysterious* *because we can substitute it for the divergent infinite series. Now the constant of the series* $1 + 1 + 1 + \&c$ *is* $-\frac{1}{2}$. See p.79 of 'Notebooks of Srinivasa Ramanujan', Volume 1, Published by TIFR, Mumbai 2012.)

1.1.2 Euler's Theorem

Critical points

Definition 1.1.2 An integer n is said to be critical for $\zeta(s)$ if both $\zeta_\infty(s)$ and $\zeta_\infty(1-s)$ are regular (i.e., no poles) at $s = n$. The set of all critical integers is called the critical set.

Observe that, by definition, the critical set is symmetric, i.e., invariant under $s \mapsto 1 - s$.

Proposition 1.1.3 *The critical set for $\zeta(s)$ consists of all even positive integers and all odd negative integers, i.e., critical set for*

$$\zeta(s) \;=\; \{\dots, 1-2m, \dots, -5, -3, -1\} \cup \{2, 4, 6, \dots, 2m, \dots\}.$$

Proof Let n be critical for $\zeta(s)$. This means two conditions on n:

1. $\zeta_\infty(s) = \pi^{-s/2}\Gamma(s/2)$ does not have a pole at $s = n$; exponentials are entire and non-vanishing and so this means $\Gamma(s/2)$ has no pole at n, i.e., $n/2 \notin \{\dots, -3, -2, -1, 0\}$, which means that n is not a non-positive even integer; *and*
2. $\zeta_\infty(1-s) = \pi^{-(1-s)/2}\Gamma((1-s)/2)$ does not have a pole at $s = n$; this translates to $\Gamma((1-s)/2)$ having no pole at n, i.e., $(1-n)/2 \notin \{\dots, -3, -2, -1, 0\}$, or $n \notin \{1, 3, 5, \dots\}$, which means that n is not an odd positive integer.

□

The critical values of $\zeta(s)$

The Bernoulli numbers are defined by the formal power series expansion of $z/(e^z - 1)$:

$$\frac{z}{e^z - 1} \;=\; \sum_{k=0}^{\infty} B_k \frac{z^k}{k!}. \tag{1.3}$$

Some easy values are:

$$B_0 = 1, \quad B_1 = -1/2, \quad B_2 = 1/6, \quad B_3 = 0, \quad B_4 = -1/30, \quad B_5 = 0, \dots .$$

Indeed, we have

$$B_{2k+1} = 0 \text{ for } k \geq 1.$$

Theorem 1.1.4 *The critical values for $\zeta(s)$ are given by:*

1. *The critical values to the right of the centre of symmetry:*

$$\zeta(2m) = \frac{(-1)^{m+1}(2\pi)^{2m}B_{2m}}{2(2m)!}.$$

2. *The critical values to the left of the centre of symmetry:*

$$\zeta(1-2m) = -\frac{B_{2m}}{2m}.$$

See Neukirch [Ne99, Chapter VII, Section 1] for a detailed proof.

Remark 1.1.4.1 Let us note the special case $\zeta(-1) = -1/12$ was 'proved' by Euler (and later rediscovered by Ramanujan) via the following intriguing calculation:

$$\begin{aligned} S &= 1+2+3+4+5+6+\cdots \\ 4S &= 4+8+12+\cdots \\ -3S &= 1-2+3-4+5-6\cdots = \frac{1}{(1+1)^2} = 1/4 \quad \Longrightarrow S = -1/12. \end{aligned}$$

The non-critical values of $\zeta(s)$

The non-critical values of $\zeta(s)$ are its values at non-critical integers, i.e., the values $\{\zeta(2m+1) : m \geq 1\}$ and $\{\zeta(-2m) : m \geq 1\}$. *The values at the odd positive integers are mysterious and the purpose of this workshop is to understand these values via the Bloch–Kato conjectures.* However, the values at the negative even integers are trivial:

Lemma 1.1.5 (Trivial zeros) *For any integer $m \geq 1$, $\zeta(s)$ has a simple zero at $s = -2m$.*

Proof Put $s = -2m$ into the functional equation to get:

$$\pi^m\,\Gamma(-m)\,\zeta(-2m) = \pi^{(1+2m)/2}\,\Gamma\left(\frac{1+2m}{2}\right)\zeta(1+2m).$$

The right hand side is finite and non-zero, therefore so is the left hand side; but $\Gamma(-m)$ is a simple pole, hence $\zeta(-2m)$ is a simple zero. □

What is mysterious about $\zeta(s)$ at $s = -2m$ is not so much the value, but rather the leading term:

$$\zeta^*(-2m) := \lim_{s\to -2m} \zeta(s)(s+2m).$$

The mystery about $\zeta(2m+1)$ is equivalent, via the functional equation, to the mystery about $\zeta^*(-2m)$. We mention the following transcendental statements only for the sake of completeness:

1. $\zeta(3)$ is irrational. (See Apéry [Ap79].)
2. The $\mathbb{Q}$-span of $\{\zeta(2m+1) : m \geq 1\}$ is an infinite-dimensional subspace of $\mathbb{R}$, i.e., infinitely many of the zeta values at odd positive integers are $\mathbb{Q}$-linearly independent. (See Ball and Rivoal [BR01].)

1.1.3 Borel's Theorem

Tamagawa number of $\mathrm{SL}_n/\mathbb{Q}$ and ζ-values

Let $n \geq 2$. Fix an isomorphism

$$\text{Top exterior} \;:\; (\mathfrak{sl}_n)/\mathbb{Z} \;\to\; \mathbb{Z}.$$

This induces the Tamagawa measure on $\mathrm{SL}_n(\mathbb{A})$ as follows: The measure dg on $\mathrm{SL}_n(\mathbb{A})$ is the product of local measures $dg = \prod_v dg_v$ and locally everywhere dg_v is the Haar measure determined by the above isomorphism. By definition, the Tamagawa number of $\mathrm{SL}_n/\mathbb{Q}$ is:

$$\tau(\mathrm{SL}_n/\mathbb{Q}) \;:=\; \mathrm{vol}\,(\mathrm{SL}_n(\mathbb{Q})\backslash \mathrm{SL}_n(\mathbb{A}))\,.$$

Theorem 1.1.6 *The Tamagawa number of* $\mathrm{SL}_n/\mathbb{Q}$ *is* 1*, i.e.,* $\tau(\mathrm{SL}_n/\mathbb{Q}) = 1$.

See, for example, Weil [We58].

Corollary 1.1.7

$$\prod_{m=2}^{n} \zeta(m) \;=\; \mathrm{vol}\,(\mathrm{SL}_n(\mathbb{Z})\backslash \mathrm{SL}_n(\mathbb{R}))\,.$$

Proof The strong approximation theorem gives:

$$\mathrm{vol}\,(\mathrm{SL}_n(\mathbb{Q})\backslash \mathrm{SL}_n(\mathbb{A})) \;=\; \mathrm{vol}\,(\mathrm{SL}_n(\mathbb{Z})\backslash \mathrm{SL}_n(\mathbb{R})) \prod_p \mathrm{vol}(\mathrm{SL}_n(\mathbb{Z}_p));$$

the left hand side is 1 by the above theorem, and for the right hand side we have

$$\mathrm{vol}(\mathrm{SL}_n(\mathbb{Z}_p)) \;=\; \prod_{m=2}^{n} (1 - p^{-m}),$$

where all the volumes are with respect to the Tamagawa measures. □

It is rather piquant to note that $\zeta(3)\pi^2/6 = \mathrm{vol}\,(\mathrm{SL}_3(\mathbb{Z})\backslash \mathrm{SL}_3(\mathbb{R}))\,.$

Statement of Borel's results

In this section we give a very brief sketch of some results of Borel [Bo77]. The serious reader should consult Borel for all details. Consider the following diagram of cohomology groups:

$$\begin{array}{ccc} H^\bullet(\mathrm{SU}(n);\,\mathbb{C}) & \xrightarrow{\quad\mu^\bullet\quad} & H^\bullet(\mathrm{SL}_n(\mathbb{R})/\Gamma_n;\,\mathbb{C}) \\ & {\scriptstyle\alpha^\bullet}\nwarrow \qquad \nearrow{\scriptstyle\beta^\bullet} & \\ & H^\bullet(\mathfrak{sl}_n(\mathbb{C});\,\mathbb{C}) & \end{array}$$

where Γ_n is an arithmetic torsion-free subgroup of $\mathrm{SL}_n(\mathbb{Z})$. The morphisms $\alpha^\bullet$ and $\beta^\bullet$ are defined in terms of invariant forms and $\alpha^\bullet$ is an isomorphism; now define $\mu^\bullet := \beta^\bullet \circ {\alpha^\bullet}^{-1}$.

All the cohomology groups in sight are exterior algebras and we can talk of indecomposable elements. (Let A be an algebra over a field which is graded by subspaces $\{A^p\}_{p\in\mathbb{N}}$. The space of indecomposable elements of A of degree p, denoted $I^p(A)$, is defined to be the quotient of A^p by the subspace of decomposable elements, i.e., subspace generated by products of elements of degree less than p.)

We can also work with rational cohomology, and let us record the following results of Borel: For an even positive integer m, the spaces $I^{2m+1}(\mathfrak{sl}_n(\mathbb{Q});\,\mathbb{Q})$ and $I^{2m+1}(\mathrm{SU}(n);\,\mathbb{Q})$ are 1-dimensional $\mathbb{Q}$-subspaces of the ambient complex spaces. (For brevity, we have let $I^{2m+1}(\mathfrak{sl}_n(\mathbb{Q});\mathbb{Q})$ to stand for $I^{2m+1}(H^\bullet(\mathfrak{sl}_n(\mathbb{Q});\mathbb{Q}))$, etc.) Borel studied the effect of the maps $\alpha^\bullet$ and $\beta^\bullet$ on these one-dimensional lines, and proved

$$\alpha^\bullet(I^{2m+1}(\mathfrak{sl}_n(\mathbb{Q});\,\mathbb{Q})) \;=\; (\pi i)^{m+1} I^{2m+1}(\mathrm{SU}(n);\,\mathbb{Q}). \tag{1.4}$$

See [Bo77, Proposition 5.4]. Similarly, one has

$$\beta^\bullet(I^{2m+1}(\mathfrak{sl}_n(\mathbb{Q});\,\mathbb{Q})) \;=\; \zeta(m{+}1)\, I^{2m+1}(\mathrm{SL}_n(\mathbb{R})/\Gamma_n;\,\mathbb{Q}), \quad (n > 8m{+}5). \tag{1.5}$$

This involves a calculation of integrating a top-degree rational form on a 'modular symbol' $H/(\Gamma\cap H) \hookrightarrow G/\Gamma$ where H is a suitable $\mathrm{SL}_1(D)$ inside $G = \mathrm{SL}_n$; this is a Tamagawa number calculation, the simplest case of which is briefly described in 1.1.3. For more details, see Borel's [Bo77, Théorème 5.5] and its proof.

The heart of Borel's paper is to construct and analyze a certain canonical morphism in a relative context (i.e., mod-maximal-compact) for real cohomology:

$$j_\Gamma^\bullet \;:\; H^\bullet(\mathrm{SU}(n)/\mathrm{SO}(n);\mathbb{R}) \;\longrightarrow\; H^\bullet(\mathrm{SL}_n(\mathbb{Z})\backslash\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n);\mathbb{R}). \quad (1.6)$$

Observe that the right hand side is also group cohomology:

$$H^\bullet(\mathrm{SL}_n(\mathbb{Z})\backslash\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n);\mathbb{R}) \;\simeq\; H^\bullet(\mathrm{SL}_n(\mathbb{Z});\mathbb{R}).$$

The main result of Borel, stemming from (1.4), (1.5) and (1.6), is the following:

Theorem 1.1.8 *Let $m = 2r$ be an even positive integer and let $n > 8m+5$. Then*

$$j_\Gamma^\bullet(I^{4r+1}(\mathrm{SU}(n)/\mathrm{SO}(n);\,\mathbb{Q})) \;=\; \frac{\zeta(2r+1)}{\pi^{2r+1}}\, I^{4r+1}(\mathrm{SL}_n(\mathbb{Z});\,\mathbb{Q}).$$

See [Bo77, Théorème 6.2]. This result has an interpretation in terms of K-groups, which we discuss in the next subsection after introducing K-groups.

1.1.4 K-groups of $\mathbb{Z}$

For us, K_m is a functor from the category of commutative rings to the category of abelian groups; Sujatha's lectures in this workshop go more deeply into algebraic K-theory that is necessary to study the Riemann ζ-function. One calls $K_0(R)$ the *projective module group* and it is defined to be the quotient of the free abelian group on $[P]$ where P runs over isomorphism classes of a finitely generated projective module by the normal subgroup generated by the relations $[P\oplus Q]-[P]-[Q]$. Since $\mathbb{Z}$ is a PID, and every finitely generated projective module is free, we get $K_0(\mathbb{Z}) = \mathbb{Z}$. The group $K_1(R)$ is called *the Whitehead group* and it is defined to be the quotient $\mathrm{GL}(R)/E(R)$ where $\mathrm{GL}(R) := \varinjlim_n \mathrm{GL}_n(R)$, the limit taken over the maps $\mathrm{GL}_n(R) \to \mathrm{GL}_{n+1}(R)$ given by $g \mapsto \mathrm{diag}(g,1)$; and $E(R)$ is the subgroup generated by all elementary matrices. Since we have taken R to be commutative, the determinant homomorphism is defined and one has $K_1(R) \cong R^\times \oplus (\mathrm{SL}(R)/E(R))$. The group $\mathrm{SL}(R)/E(R)$ is often denoted $\mathrm{SK}_1(R)$ and is called the reduced Whitehead group. If R is a Euclidean domain, one knows that $\mathrm{SK}_1(R) = \{1\}$. Hence $K_1(\mathbb{Z}) = \mathbb{Z}^\times \cong \mathbb{Z}/2$. (See, for example, Milnor [Mi71].) For a ring R, one defines

$$K_m(R) \;:=\; \pi_m(BGL(R)^+), \quad (1.7)$$

i.e., the m-th homotopy group of Quillen's plus construction applied to the classifying space of the limit $\mathrm{GL}(R)$ of general linear groups.

Computing K-groups is a highly non-trivial problem, and even for K-groups of $\mathbb{Z}$ not everything is known. In the next subsection we give a summary of some more precise results on $K_m(\mathbb{Z})$.

What is known about $K_m(\mathbb{Z})$?

The following brief summary on $K_m(\mathbb{Z})$ is taken from Weibel [We05]. We begin with two general finiteness results:

1. $K_m(\mathbb{Z})$ is a finitely generated abelian group. (Quillen [Qu73].)
2. Rank of $K_m(\mathbb{Z})$ is 1 if $m \geq 5$ is 1 mod 4; in all other cases $K_m(\mathbb{Z})$ is a finite group, i.e., has rank 0. (Borel [Bo74].)

$$
\begin{array}{lcl@{\qquad}lcl}
K_0(\mathbb{Z}) & = & \mathbb{Z} & K_{8a}(\mathbb{Z}) & = & 0? \\
K_1(\mathbb{Z}) & = & \mathbb{Z}/2 & K_{8a+1}(\mathbb{Z}) & = & \mathbb{Z}\oplus\mathbb{Z}/2 \\
K_2(\mathbb{Z}) & = & \mathbb{Z}/2 & K_{8a+2}(\mathbb{Z}) & = & \mathbb{Z}/2c_{2a+1} \\
K_3(\mathbb{Z}) & = & \mathbb{Z}/48 & K_{8a+3}(\mathbb{Z}) & = & \mathbb{Z}/2w_{4a+2} \\
K_4(\mathbb{Z}) & = & 0 & K_{8a+4}(\mathbb{Z}) & = & 0? \\
K_5(\mathbb{Z}) & = & \mathbb{Z} & K_{8a+5}(\mathbb{Z}) & = & \mathbb{Z} \\
K_6(\mathbb{Z}) & = & 0 & K_{8a+6}(\mathbb{Z}) & = & \mathbb{Z}/c_{2a+2} \\
K_7(\mathbb{Z}) & = & \mathbb{Z}/240 & K_{8a+7}(\mathbb{Z}) & = & \mathbb{Z}/w_{4a+4}
\end{array}
\tag{1.8}
$$

The question marks mean that it is expected $K_{4a}(\mathbb{Z}) = 0$. This is proven for $a = 1$ and is open as yet for $a \geq 2$. The numbers c_m and w_m are defined as follows:

$$c_m \;=\; \text{numerator of } (-1)^{m+1}B_{2m}/4m. \tag{1.9}$$

(It is understood that if we talk of the numerator a of a rational number a/b then one has taken the rational to be in its lowest form, i.e., a and b are relatively prime.) Let $\mathcal{W}$ be the group of all roots of unity in $\overline{\mathbb{Q}}$. Then $\mathcal{W}$ is naturally a $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-module, since if $w \in \mathcal{W}$ and $g \in G_{\mathbb{Q}}$ then $g(w) \in \mathcal{W}$. For any integer $m \geq 1$ we let $\mathcal{W}(m)$ stand for the $G_{\mathbb{Q}}$-module where $g \in G_{\mathbb{Q}}$ acts on $w \in \mathcal{W}(m) = \mathcal{W}$ via $g \cdot_m w := g^m(w)$. One says $\mathcal{W}(m)$ is the Galois module $\mathcal{W}$ with a Tate twist by m. Now define

$$w_m \;:=\; |\{w \in \mathcal{W} \;:\; g^m(w) = w,\ \forall g \in G_{\mathbb{Q}}\}|, \tag{1.10}$$

i.e., it is the cardinality of the set of those roots of unity which are fixed by $G_{\mathbb{Q}}$ under the m-twisted action.

Borel regulators and non-critical values

Let us go back to Theorem 1.1.8: recall that $m = 2r$ is an even positive integer and $n \gg m$, then

$$j_\Gamma^\bullet(I^{2m+1}(\mathrm{SU}(n)/\mathrm{SO}(n);\,\mathbb{Q})) \;=\; \frac{\zeta(m+1)}{\pi^{m+1}}\,I^{2m+1}(\mathrm{SL}_n(\mathbb{Z});\,\mathbb{Q}).$$

Now pass to the limit over n. Define $X_u := \varinjlim_n \mathrm{SU}(n)/\mathrm{SO}(n)$ and $\mathrm{SL}(\mathbb{Z}) := \varinjlim_n \mathrm{SL}_n(\mathbb{Z})$. Then it is known that we have a duality:

$$I^{2m+1}(X_u;\mathbb{Q}) \;\times\; (\pi_{2m+1}(X_u)\otimes_{\mathbb{Z}}\mathbb{Q}) \;\longrightarrow\; \mathbb{Q},$$

and similarly,

$$I^{2m+1}(\mathrm{SL}(\mathbb{Z});\mathbb{Q}) \;\times\; (K_{2m+1}(\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Q}) \;\longrightarrow\; \mathbb{Q}.$$

(Recall that $m = 2r$ is even, and so $K_{2m+1}(\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Q}$ is a one-dimensional $\mathbb{Q}$-vector space.) Fix a basis x_m^* for $\pi_{2m+1}(X_u)$, and y_m^* for $K_{2m+1}(\mathbb{Z})$. Let x_m and y_m be the dual basis.

Definition 1.1.9 (Borel Regulators)

$$j_\Gamma^\bullet(x_m) \;=\; R_m(\mathbb{Q})\,y_m.$$

From Theorem 1.1.8 and Definition 1.1.9 we get the following beautiful result of Borel on the non-critical values of Riemann zeta function:

Theorem 1.1.10 (Borel) *Let $r \geq 1$. Then*

$$\frac{\zeta(2r+1)}{\pi^{2r+1}} \;\sim\; R_{2r}(\mathbb{Q}),$$

where $\sim$ means up to a non-zero rational number.

1.1.5 Lichtenbaum's conjecture

Critical values and K-groups

Reference: Lichtenbaum [Li73, Conjecture 2.4].

Theorem 1.1.11 (Critical values on the left) *Up to 2-torsion for any odd integer $m \geq 1$*

$$|\zeta(-m)| \;=\; \frac{|K_{2m}(\mathbb{Z})|}{|K_{2m+1}(\mathbb{Z})|}.$$

Proof Follows from Theorem 1.1.4 and (1.8). □

The caveat 'up to 2-torsion' is necessary as can already be seen from the case $m = 1$:

$$\zeta(-1) \;=\; -\frac{B_2}{2} \;=\; -\frac{1}{12},$$

whereas $|K_2(\mathbb{Z})| = 2$ and $|K_3(\mathbb{Z})| = 48$ giving

$$\frac{|K_2(\mathbb{Z})|}{|K_3(\mathbb{Z})|} \;=\; \frac{1}{24}.$$

K-theoretic Lichtenbaum's conjecture

In [Li73, Question 4.2] Lichtenbaum formulated a conjecture for ζ-values at any negative integer $-m$ in terms of certain higher regulators $R'_m(\mathbb{Q})$ which are essentially the same as Borel's regulators $R_m(\mathbb{Q})$. For Lichtenbaum's definition of $R'_m(\mathbb{Q})$ see [Li73, p.498–499].

Conjecture 1.1.12 (Any special value on the left) For any integer $m \geq 1$, possibly up to 2-torsion, we have

$$\zeta^*(-m) \;=\; \pm\frac{|K_{2m}(\mathbb{Z})|}{|K_{2m+1}(\mathbb{Z})_{\text{tors}}|} \cdot R'_m(\mathbb{Q}).$$

A slightly modified version of Conjecture 1.1.12 was proved for any abelian number field by Kolster, Nguyen Quang Do and Fleckinger [KNF96, Theorem 6.4]; some errors on Euler factors in [KNF96] have been corrected in [BN02]. For a result character by character, see [HK03]. The reader is also referred to the survey article by Flach [Fl04] for a historic introduction.

Cohomological Lichtenbaum's conjecture

The K-theoretic Lichtenbaum's conjecture can be restated in the language of Galois cohomology. The connection is provided by the Quillen–Lichtenbaum conjecture which relates K-groups to Galois cohomology. (The reader should look at Lichtenbaum [Li73, Conjecture 2.5] and Huber–Kings [HK03, p.410].) Indeed, the formulas in (1.8) cited from [We05] use this connection between K-groups and Galois cohomology.

Give Spec($\mathbb{Q}$) the étale topology. Then the category of discrete $G_{\mathbb{Q}}$-modules is equivalent to the category of sheaves of abelian groups over Spec($\mathbb{Q}$), and Galois cohomology is the same as sheaf cohomology which in this case is called étale cohomology. For any $m \geq 1$, the $G_{\mathbb{Q}}$-module $\mathcal{W}(m)$ gives a sheaf on Spec($\mathbb{Q}$). Denote this sheaf also by $\mathcal{W}(m)$. Fix a prime ℓ. Let $X_\ell \;:= \text{Spec}(\mathbb{Z}[1/\ell])$, and let

$$j \;:\; \text{Spec}(\mathbb{Q}) \hookrightarrow X_\ell$$