CHAPTER I

THE LAWS OF ATMOSPHERIC MOTION

Another storm a'brewing—I hear it sing i' the wind *Tempest*, Act II, Sc. 2.

NEWTON'S LAWS OF MOTION

THE problem of dynamics as applied to the atmosphere is to trace the relation between the motion of the air as observed in the winds at the surface by anemometers, in the upper air by aeroplanes, or as inferred from the motion of clouds or balloons, and the "forces" which are assumed to be operative in producing or in changing the motion. For the solution to be accepted the relation established must be quantitative as well as qualitative, that is to say the forces which are involved must be shown to be adequate to give not merely a general idea of the phenomena which have to be explained but also an accurate measure of the motion which has been observed.

When a satisfactory relation of that kind has been established between the observed motion of the air and the forces available to produce it and maintain it, the same process can be applied to changes in the present state of motion, which will be produced by the operation of the available forces, provided these can be expressed numerically. The prediction of weather will then become a "mathematical certainty."

The classical example which mathematicians have in mind when they endeavour to trace a quantitative or numerical relation between the forces operative in the atmosphere and the motion which results therefrom is the extraordinarily accurate solution of the problem of the dynamics of the solar system as expounded by Sir Isaac Newton in his *Philosophiae Naturalis Principia Mathematica* which contains also applications of the same principles to other dynamical problems.

For more than two centuries that exposition was regarded as the outstanding example of the ingenuity of the human mind—*genus humanum ingenio superavit*—and may still be so regarded unless the modification introduced in the present century by Prof. A. Einstein on the new principle of relativity of motion be considered a still more impressive example of human genius.

The mysterious influence concealed under the name of gravity has become a spatial contortion, and we have found the new point of view useful for weather-study in relation to the other mysterious influence "entropy."

The efforts which have been made hitherto by mathematicians to solve the problem of the dynamics of the earth's atmosphere have been guided by the Principia of Newton, and based upon three laws or axioms, which express in singularly well-chosen words the general ideas of force in relation to motion. They were evolved in the seventeenth century from the observations of astronomers and the experiments of natural philosophers. In the original Latin, which hardly needs translation, they are as follows:

Axiomata, sive Leges motus

Lex I. Corpus omne perseverare in statu quo quiescendi vel movendi uniformiter in directum, nisi quatenus illud a viribus impressis cogitur statum suum mutare.

Lex II. Mutationem motus proportionalem esse vi motrici impressae, & fieri secundum lineam rectam qua vis illa imprimitur.

Lex III. Actioni contrariam semper & aequalem esse reactionem: sive corporum duorum actiones in se mutuo semper esse aequales & in partes contrarias dirigi.

It is one of the great achievements of the enunciation of the laws of motion that they embody a conception of force which can be employed in mathematical argument, namely that the measure of force, whatever name be given to it, is the rate at which the momentum of the moving body is changing. Momentum must be understood to mean a new physical quantity expressed numerically as the product of the mass of the moving body and its velocity. If the rate of change is uniform, as it is assumed to be in many illustrative examples, the force is expressed by the actual change of momentum in the unit of time—in other words the mass multiplied by the acceleration.

The conception of force

The reader will notice that he is assumed to be acquainted with what is to be understood by *vis* (force), *vires impressae* (impressed forces), *vis motrix impressa* (impressed motive force) and *actio*, *reactio*, for which the terms action and reaction are used in English. And indeed the conception of force, which is of course fundamental if forces are to be employed to calculate motion, is one which no ordinary person makes any difficulty about understanding. And yet, when the basis of the understanding comes to be examined, it is difficult to realise that in the specification of any force anything more is meant than that directly or indirectly, somehow and somewhen, the force has shown a capacity for resisting or balancing the attractive force of gravity.

It was doubtless the *brut* realism of this statement compared with the idealism of measuring force by momentum, which is never or hardly ever exactly possible, that led some distinguished engineers and mathematicians to insist upon measuring forces in terms of that of gravity upon a pound, and to turn a scornful lip towards the poundal and the foot-poundal for measuring force and work, which were offered by the advocates of systematic international measurement as a concession to British prejudice in such matters.

Let us note that, in order to be consistent in estimating force, the motion must be referred to the proper centre, otherwise we may arrive at paradoxical conclusions. For example, consider two independent planets, masses M and m, acting upon one another with some force like gravity, and consider the motion of each as observed from the other. Dealing with a period of time so short that the variation in the intensity of the force is insignificant the changes in their velocity in unit time are V and v. The force on the one is MV and the force on the other mv. But since velocity is relative the approach of M to m is the same as the approach of m to M; hence v and V are equal; but by Law III the action MV is equal to the reaction mv. Hence the two masses must be equal—which does not appear in the original hypothesis. The velocity measured ought not to be that of one body relative to the other but of each with regard to the "centre of mass" of the two.

The principles of Newton in the hands of mathematicians have been found serviceable in the solution of many dynamical problems relating to the heavenly bodies, the earth and its figure, the sea and its tides, the air and its pressure; in particle dynamics, rigid dynamics and hydrodynamics; and the feeling is quite general among mathematicians that the line of approach to the solution of the general meteorological problem is by the mathematical evaluation of the effect of known forces upon the state of the atmosphere which can be regarded as initial. In the case of the atmosphere the question that arises is whether we really know all the forces which are operating to produce and maintain the motion which may be observed in a sample of air.

It is this practice of assuming a knowledge of the forces in order to compute the motion instead of using the observed motion in order to infer the forces that constitutes the difference between the deductive, or mathematical, and the inductive, or observational method of treating the subject, either of which may furnish appropriate material for this volume.

The recognised forces

What then, let us ask, are the forces which meteorologists can offer to mathematicians for their enterprise? Clearly gravity, the great Newtonian force, is one, of which the character and magnitude are quite well understood even if its origin has still to be accounted for.

The latest view of mathematical physicists, as expressed by Einstein, seems to be that the force of attraction, or the change of momentum which is its equivalent, is an affection of space in the neighbourhood of material objects such as the bodies of the solar system; and it may be worth the reader's while to think of that mode of explanation in relation to the restriction of the motion of air to an isentropic surface which we have set out in chap. VI of vol. III.

Secondly there is centrifugal force which is perhaps not really a Newtonian force at all, and yet comes naturally to the mind when one thinks of a heavy bob, as that of a pendulum, whirled round one's head at the end of a string. Something is wanted to account for the pull of the bob on the string, and if we say that the pull, which requires the tension of the string to balance it, is centrifugal force we can urge in justification that on occasion it may be more than the string can bear and it is centrifugal force which produces disruption. If not, what does? In the atmosphere a centrifugal force which is never absent except at the very poles is that due to the rotation of the earth. It takes a hand in every meteorological phenomenon by pushing sideways any air that moves.

Next pressure, the statistical expression of momentum transferred by the impact of the molecules of a parcel of air upon its boundaries, in practice only a special manifestation of the force of gravity. If we consider the motion of the air we regard the bombardment of any part of its boundary by the molecules of its environment as producing a force normal to the surface bombarded, measured by the product of the pressure and the area of the surface, and balancing the pressure of the interior.

Fourthly—flotation, or, as commonly used in meteorology, convection, on the Archimedian principle of an upward force, equal to the weight of fluid displaced, acting upon any body immersed in a fluid, this also is another effect of gravity.

Fifthly—surface tension, the expression of what is called capillarity, which moulds water-drops into spherical shape as we have seen in vol. III. The peculiarities of this form of force are held to be responsible for the dynamical manifestations of thunderstorms, but it operates only in the surfaces of liquids, unless the colloidal properties of mixtures of foreign bodies with air may be in some way associated with it.

Sixthly—the force of turbulence, the tendency of moving air when it is passing other air with sufficient rapidity to roll itself up into something suggestive of a whirl or vortex. We shall have something to say later about the vitality of this particular mode of motion.

Seventhly—friction, a mysterious force which intrudes itself whenever and wherever the surfaces of bodies in contact slide, or even tend to slide, one past the other; and which, so far as the atmosphere is concerned, has something to do with viscosity—of which later.

There are other forces which have sometimes to be taken into account by those who would refer to Newton's laws of motion as embodying the principles upon which an answer must be found to the oft-repeated question, "Will it rain to-morrow?" But the seven examples which we have cited are sufficient to indicate that the problem is liable to a good deal of complication.

The classical mathematical method of dealing with any dynamical problem is to write down equations representing the balance between changes of motion (in terms of momentum), regarded as unknown, and impressed forces, regarded as known when normal information is available.

Natural philosophers have not always waited for that stage; they have on occasions proposed instead such approximations to a solution as can be obtained by general reasoning from their knowledge of the nature and distribution of the operative forces.

As we have already seen (vol. I, p. 288), that kind of reasoning provided an explanation of trade-winds and monsoons, of land- and sea-breezes, based upon general considerations of thermal convection and the rotation of the earth. Such explanations appeared for many generations in the text-books of physical geography; but there was no attempt to obtain numerical values either of the direction or of the velocity of the winds or their variation from season to season. In such questions however it is unsafe to disregard any influence or assume a complete knowledge of the causes, and so long as there is any uncertainty about the causes it is fair to say that for nearly all purposes an accurate description of the phenomena is the best substitute for an explanation.

Let us accordingly pass in review the steps which have been taken towards the numerical expression of the forces which we have enumerated.

1. **Gravity**—strictly speaking the force of attraction between all material substances in the universe, which in the case under our consideration would add to any mass, free to move, a velocity of approximately 981 cm/sec in every second, directed towards the centre of gravity of the earth. In practice however the gravity denoted by the letter g takes account also of the earth's rotation and is the acceleration of any body, free to move, along the "vertical," that is to say at right angles to the "horizontal" or "level" surface which forms the conventional boundary of a fluid earth. It points to the centre of gravity of the earth only at the equator and at the pole; elsewhere in the northern hemisphere it points south of the centre of gravity, in the southern hemisphere to the north of it.

Its numerical expression for a point on the earth's surface is given by the equation:

$g = 980{\cdot}617\,(1 - {\cdot}00259 \cos 2\phi)\,(1 - 5z/4\epsilon)$ c, g, s units, where ϕ is the latitude, z the height above sea-level, and ϵ the earth's radius

$= 980{\cdot}617\,(1 - {\cdot}00259 \cos 2\phi)\,(1 - 1{\cdot}96 \times 10^{-7}z)$, where z is the height in metres.

[ϵ is chosen as a symbol for the earth's radius because it comes near to being a semicircle with a radius from its middle point.]

This formula takes into account the additional attraction of the high ground and supposes the mean density of the elevated area to be equal to one-half of the mean density of the earth.

980·617 is the value of the gravitational acceleration in c, g, s units at sea-level in latitude 45°. For the determination of gravity at points above the earth's surface, the factor $1/(1 + z/\epsilon)^2$, which equals approximately $(1 - 2z/\epsilon)$, replaces $(1 - 5z/4\epsilon)$.

(*Computer's Handbook, Introduction*, M.O. 223, 1921, p. 9.)

2. **Centrifugal force.** The numerical expression for the effect produced at a point on the earth's surface by the rotation of the earth is represented by an acceleration $\omega^2\epsilon \cos\phi$ outwards, perpendicular to the axis of rotation, where ω is the angular velocity of the earth and $\epsilon \cos\phi$ the distance of the point affected from the axis of rotation. We have seen that the value of g takes account of the earth's rotation. The vertical component gives a force $\epsilon\omega^2 \cos^2\phi$ in diminution of g. The horizontal component gives a force $\epsilon\omega^2 \cos\phi \sin\phi$ directed towards the equator. This is balanced by the inclination of the earth's surface to the surface of a true sphere of the same mass and volume as the earth. The geometrical slope towards the pole which is represented by the excess of the equatorial radius (6377 km) over the polar radius (6356 km), and which therefore gives a greater distance from the earth's centre at the equator than at the poles, is known as the geoidal slope (vol. III, p. 296). It does not mean that a body on a perfectly smooth sea actually drifts from the equator to the pole, but it would so drift (and the water too) if the earth's rotation should by any chance ease off.

3. **Pressure.** We have so often used the expression of pressure in previous volumes that here we need only remind the reader that in computing pressure we assume a quiescent atmosphere (vol. III, p. 215) and express the pressure numerically with the aid of Laplace's equation $dp = -g\rho dz$. In considering the motion of an element of fluid it is the difference of pressure on two opposite sides that counts as the *vis impressa* upon the element.

4. **Convection.** This likewise has been in familiar use all through the work of the three previous volumes. Chapters VIII and X in vol. III have already been devoted to it. The numerical expression for the force of flotation of a volume v of air of density ρ in an environment of density ρ' is $\iiint g\,(\rho' - \rho)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$ or, if local variation of density is negligible, $g\,(\rho' - \rho)\,v$.

5. **Capillarity.** We have already explained that capillarity only comes in when water-drops or other fluid bodies are under consideration. We may refer to the chapter on the subject in Maxwell's *Theory of Heat*, or any other text-book of physics, for the details of the numerical expression of the force. The effect of capillarity is expressed as a tension in the surface as though it were made of flexible material. Its numerical value is expressed by the force per unit of length of a line drawn in the surface. The tension of the surface between water and air is 74 dynes per cm, and of that between mercury at 290·5tt (17·5° C) and air 547 dynes per cm.

6. **Diffusive forces: viscosity and turbulence.** These forces cannot be evaluated like the five which we have considered. Their numerical expression is a statistical one and is derived from a consideration of experiments on diffusion in which the progressive distribution over one part of space, of matter or energy drawn from an adjacent part of space, is watched.

7. **Friction.** Here we have to distinguish between viscosity which gives the equivalent of a frictional or tangential force between two parts of a fluid moving one past the other and the friction between a fluid and a solid surface over which it is moving. The former may be regarded as subject to a general law for which a coefficient of viscosity is appropriate, the latter depends not only on the viscosity of the fluid but also on the nature of the solid surface.

CONSERVATION OF MASS AND ENERGY

The numerical expression of the diffusive and frictional forces of the atmosphere requires a more formal introduction than the familiar forces of gravity, centrifugal action, pressure and convective force; but before dealing with that part of the subject we may remind the reader that the whole of the calculus of weather which we have in view accepts as primary conditions the laws of conservation of mass and conservation of energy as explained in the introductory paragraphs of the chapter on "Air as Worker" (chap. VI, vol. III). We must include these conditions among the laws which govern the movements of the atmosphere. Let us therefore state them.

Law IV. Conservation of mass. In the computation of any atmospheric movement the expression of the distribution of the mass of the moving parts must account for any changes which may take place in the boundaries of a selected parcel and the mass contained within them on the understanding that the total mass of the whole system, viz the moving air and its environment, is unalterable.

The expression of this principle of the conservation of mass is referred to the coordinates in which the position of any parcel is expressed and will be included as the "equation of continuity" in our setting out of the general equations of motion of a parcel of air in the free atmosphere.

In considering the behaviour of "Air as Worker" in chap. VI of vol. III we have not paid any special attention to the particular forms of "work" which the air may perform, whether the transference of energy expressed by work results in the kinetic energy of moving mass, or any other of the forms which we have enumerated; but in the dynamics of the atmosphere these differences in the forms of energy are precisely the subjects of study. It will be sufficient here if we quote the statement of the principle as expressed by Maxwell.

Law V. Conservation of energy. The total energy of any material system is a quantity which can neither be increased nor diminished by any action between the parts of the system, though it may be transformed into any of the forms of which energy is susceptible.

This principle indeed is so vital to the study of the dynamics of a material system that it can be used as the starting-point for the expression of the equations of motion following the method of Lagrange to which reference will be made in chap. II.

What exactly is the material system under which the energy of the movements of the atmosphere can be studied requires a little consideration because it must take account of all the operating forms of energy of which gravity and solar radiation are the most important. But perhaps it will be sufficient if we regard the rotating earth with its atmosphere as the material system, allowing for radiation as energy supplied from space or lost thereto.

CONSERVATION OF MOMENTUM, LINEAR AND ANGULAR

While we are dealing with questions of conservation we must remember certain conditions relating to momentum.

A particular form of conservation is implied in Newton's third law. The equality of action and reaction between two bodies requires that if the action and reaction are measured by change of momentum, in any case of the influence of one body on the other the gain of momentum by the one body corresponds with the loss of momentum in the same direction by the other. Hence the momentum and indeed the component of momentum in any given direction is conserved during the dynamical operation of one body on another.

We must understand that the measure of the motion is duly taken with regard to the common centre of gravity of the two, otherwise we get into the difficulty suggested on p. 2.

This form of conservation is a notable matter because the two bodies regarded as a system may lose energy while momentum is conserved.

For example, two bodies of equal mass impinging one on the other with equal velocities with reference to their common centre will have the kinetic energy of both annihilated during the transference of momentum on impact, and lost unless the heat equivalent of the energy is brought into account. For this form of energy the laws of motion make no allowance.

There is another form of conservation of momentum which can be deduced from the laws of motion, namely that of angular momentum (see p. 45). That also has interesting aspects from the point of view of energy.

The angular momentum or moment of momentum of a moving mass with regard to a point in the plane of its motion is measured by the product of the momentum mv and the distance s of the line of momentum from the point. It is therefore represented algebraically by mvs and geometrically by twice the area of the triangle formed by joining to the point of reference the extremities of a line representing the momentum.

If the force under the action of which the mass is moving is a central force, that is if it always passes through the point of reference, it can never produce any acceleration at right angles to itself, and therefore though the velocity may change the moment of momentum is not affected. It remains constant throughout the motion. The motion will be such that the area of the triangle formed by joining the point of reference to the extremities of the line representing the velocity will be constant.

This is the expression of the law of equal areas, known as Kepler's first law of the motion of the planets with reference to the sun.

What is true of a planet revolving round the sun is equally true of a ring of particles rotating about a centre, and therefore true for a ring of air, or part of a ring, rotating about the earth's axis.

It is a property of great importance in the study of atmospheric motion, and we therefore enunciate:

Law VI. *Conservation of angular momentum.* Any portion of a ring of air rotating about the earth's axis under the influence of forces which are directed to or from the axis will conserve its moment of momentum or angular momentum.

An experiment in illustration of this law is described by Aitken (see p. 256).

THE LAW OF DIFFUSION

Let us now consider in greater detail the nature of the diffusive forces. In the illustration of the superior mirage on p. 61 of vol. III we have referred to gradations of density of a solution of sugar as produced by the gradual diffusion of sugar from the bottom of the vessel upwards through the water of the layers above it. The gradation may be expressed by the strength of the sugar solution (the amount of sugar per unit of volume) at different heights above the layer at the bottom. The process is expressed by an equation of the type $\frac{d\theta}{dt} = \mu \frac{d^2\theta}{dx^2}$, derived by J. B. J. Fourier for the diffusion of heat by conduction in a bar of metal and known by some as Fourier's equation and by others as Fick's equation.

The diffusion may be the diffusion of heat (conduction), the diffusion of a salt through water, or of water-vapour through air, or the diffusion of momentum between two streams of air with different velocities (viscosity), or the diffusion of potential temperature (entropy) by turbulence. All these processes are reduced to one form of expression by the consideration that the quantity of the element which diffuses across any area is proportional to the change in the strength of the element along its path. The coefficient of proportionality is known as the coefficient of diffusion. Hence at any point of its progress the rate of change (dm/dt) in the strength of the diffusing element is proportional to the space rate of change of the gradient of the element along the line of travel. In algebraical form the law is expressed as $\frac{dm}{dt} = \mu \frac{d^2m}{dx^2}$, where μ is a constant different for each of the different diffusive processes.

Thus we may enunciate as a law which takes its part in the control of atmospheric motion:

Law VII. In an atmosphere stratified in layers, when the rate of change of an element with respect to time is proportional to the space rate of change of the gradient of the element in the direction of flow, the element is said to be diffusing.

The simplest case is that of conduction of heat through a body of homogeneous material in which the diffusing element may be taken to be the energy expressed by temperature (vol. III, p. 223), and we can consider the flow of heat from one side of a conducting plate to the other.

Viscosity[1]

In the same way we can treat viscosity which is used in the expression of the force at a surface of separation in a stream of air the layers of which are in relative motion. The average momentum of the molecules of air is greater in a layer of greater velocity and the force arises, as we have said in chap. VIII of vol. III, from the exchange of mass between two layers in contact in consequence of the inherent velocity of the movement of the molecules of which the gas is composed. The mean square of the molecular velocity of a gas is $\overline{V}^2 = 3p/\rho$, where p is the pressure, ρ the density (which must be expressed in terms of the fundamental units). Hence regarding air as a "homogeneous" gas with a density of ·001161 at 300tt, and pressure 10^6 c, g, s units, we get $\overline{V}^2 = (3 \times 10^6 \times 10^3)/1{\cdot}161$ cm²/sec² $= 2{\cdot}583 \times 10^9$; $\overline{V} = 5{\cdot}08 \times 10^4 = 50800$ cm/sec as the velocity of mean square at that temperature.

The exchange that takes place in consequence of the lively bombardment of one layer by its neighbour carries fast-moving air downwards and slow-moving air upwards and tends to equalise the momentum much in the same way as, on a larger scale, the turbulence of the flowing air produces a diurnal variation of wind-velocity as explained by Espy and Köppen, see p. 96.

The effect of viscosity in a stream of air, the consecutive layers of which show velocity increasing at the rate dV/dz per unit of distance z across the stream, is a retarding force F opposite to V acting upon each unit area of the faster moving layer such that $F = -\mu dV/dz$, where μ is called the coefficient of viscosity.

The numerical value of μ for air at 0° C in c, g, s units is 0·000168. Hence in a horizontal air-current which increases in the vertical at 10 m/sec per km of height, or ·01 cm/sec per cm, the retarding force upon any square centimetre of any layer is $\cdot 01 \times \cdot 000168$ dynes, or $1{\cdot}68 \times 10^{-2}$ dynes per square metre; the rate of loss of momentum across that area of the upper layer is $1{\cdot}68 \times 10^{-2}$ g cm/sec².

[1] See a lecture on 'Turbulence,' by G. I. Taylor, *Q.J. Roy. Meteor. Soc.*, vol. LIII, 1927, p. 201.

The dimensions of F are M/LT² and in consequence the dimensions of μ are M/LT, that is mass per unit of length time.

So far we have supposed the *direction* of the stream V to be the same at all levels, differing only in speed in consequence of the viscous friction. That might be so in a laboratory experiment, but in the free air the earth's rotation always gives a force varying with the speed and at right angles to the motion.

To allow for that in the frictional force we must take that force as equivalent to the rate of change of momentum in the stream. F will be equal and opposite to dM/dz and therefore proportional to dV/dz.

If we can disregard variations in the density of the viscous substance and concern ourselves only with the relative motion of successive layers, it may be convenient to base the calculations on a volume-unit instead of the ordinary mass-unit. Thus we can choose the mass of unit volume of the fluid as mass-unit instead of a gramme, understanding, of course, in that case that all transference of momentum is by change of velocity without any change of density, a condition not strictly satisfied in the case of air or any other gas, but sufficiently nearly so for most practical purposes. When this mass-unit is employed, the coefficient of viscosity is called "kinematic" as distinguished from the original dynamic coefficient and is denoted by ν. The dimensions of ν are L^2/T.

To keep the viscosity equation numerically true the transfer of momentum per unit of area, which is expressed in ordinary c,g,s units as $F = -\mu dV/dz$, becomes $F' = -\nu dV/dz$, as expressed in "kinematic units" where $\nu = \mu/\rho$. In these kinematic units the unit of mass is the mass in grammes of a cubic centimetre of the air.

With viscosity measured in this kinematic fashion, force is the rate of change of the momentum of a cubic centimetre. The loss of momentum per second across a square metre of surface in the case quoted above for air of density ·00125 g/cc is $800 \times 1.68 \times 10^{-2}$ or 13·44 cm²/sec. This expresses the transfer of momentum from an upper surface to a lower one across a layer in which the rate of change of velocity and consequently the transfer (or conduction) of momentum is uniform throughout the layer. What the upper surface loses the lower surface gains: each of the intermediate surfaces receives the same amount from the one next above and transmits the same to the one next below.

In order to study the changes in the distribution of velocity in the intervening layer, taking into account variations of velocity in the horizontal as well as in the vertical, the unit of volume is convenient.

Consider an element δx, δy, δz, with velocity V at the base and $V + \frac{\partial V}{\partial z}\delta z$ at the upper surface. The force in c, g, s units on the lower surface is $F = -\mu\,\delta x\,\delta y\,\partial V/\partial z$ and on the upper surface being opposite in direction to F is

$$-\left(F + \frac{\partial F}{\partial z}\delta z\right) = \mu\,\delta x\,.\,\delta y\,.\,\partial V/\partial z + \mu\,\delta x\,\delta y\,\frac{\partial^2 V}{\partial z^2}\delta z.$$

Hence $-\frac{\partial F}{\partial z}\delta z = \mu\frac{\partial^2 V}{\partial z^2}\delta x\,.\,\delta y\,.\,\delta z$, or if R is the force in dynamic units on the element per unit of volume $R = \mu\,\partial^2 V/\partial z^2$.