1

Galois groups of local fields, Lie algebras and ramification

Victor Abrashkin

Department of Mathematical Sciences, Durham University, Science Laboratories, South Rd, Durham DH1 3LE, United Kingdom & Steklov Institute, Gubkina str. 8, 119991, Moscow, Russia
E-mail address: victor.abrashkin@durham.ac.uk

ABSTRACT. Suppose K is a local field with finite residue field of characteristic $p \neq 2$ and $K_{<p}(M)$ is its maximal p-extension such that $\mathrm{Gal}(K_{<p}(M)/K)$ has period p^M and nilpotent class $< p$. If char $K = 0$ we assume that K contains a primitive p^M-th root of unity. The paper contains an overview of methods and results describing the structure of this Galois group together with its filtration by ramification subgroups.

Introduction

Everywhere in the paper p is a prime number. For any profinite group Γ and $s \in \mathbb{N}$, $C_s(\Gamma)$ denotes the closure of the subgroup of commutators of order s.

Let K be a complete discrete valuation field with a finite residue field $k \simeq \mathbb{F}_{p^{N_0}}$, $N_0 \in \mathbb{N}$. Let K_{sep} be a separable closure of K and $\Gamma_K = \mathrm{Gal}(K_{sep}/K)$. Denote by $K(p)$ the maximal p-extension of K in K_{sep}. Then $\Gamma_K(p) = \mathrm{Gal}(K(p)/K)$ is a profinite p-group. As a matter of fact, the major information about Γ_K comes from the knowledge of the structure of $\Gamma_K(p)$. This structure is very well known and is related to the following three cases (ζ_p is a primitive p-th root of unity) [17]:

— char $K = p$;
— char $K = 0$, $\zeta_p \notin K$;
— char $K = 0$, $\zeta_p \in K$.

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In all these cases the maximal abelian quotient of period p of $\Gamma_K(p)$ is isomorphic to K^*/K^{*p}. Therefore, $\Gamma_K(p)$ has infinitely many generators in the first case, has $[K:\mathbb{Q}_p]+1$ generators in the second case and $[K:\mathbb{Q}_p]+2$ generators in the third case. In the first two cases $\Gamma_K(p)$ is free and in the last case it has one relation of a very special form, cf. [17, 23, 24].

The above results can't be considered as completely satisfactory because they do not essentially reflect the appearance of $\Gamma_K(p)$ as a Galois group of an algebraic extension of a local field. In other words, let LF be the category of couples (K, K_{sep}) where the morphisms are compatible continuous morphisms of local fields and let PGr be the category of profinite groups. Then the functor $(K, K_{sep}) \mapsto \Gamma_K(p)$ (as well as the functor $(K, K_{sep}) \mapsto \Gamma_K$) is not fully faithful.

The situation can be cardinally improved by taking into account a natural additional structure on $\Gamma_K(p)$ and Γ_K given by the decreasing filtration of ramification subgroups. The ramification filtration $\{\Gamma_K(p)^{(v)}\}_{v\geqslant 0}$ of $\Gamma_K(p)$ (as well as the appropriate filtration $\{\Gamma_K^{(v)}\}_{v\geqslant 0}$ of Γ_K) has many non-trivial properties. For example, it is left-continuous at any $v_0 \in \mathbb{Q}$, $v_0 > 1$, i.e. $\bigcap_{v<v_0} \Gamma_K(p)^{(v)} = \Gamma_K(p)^{(v_0)}$, but is not right-continuous, i.e. the closure of $\bigcup_{v>v_0} \Gamma_K(p)^{(v)}$ is not equal to $\Gamma_K(p)^{(v_0)}$. Another example [14, 15], for any $v_1, v_2 < v_0$, $(\Gamma_K(p)^{(v_1)}, \Gamma_K(p)^{(v_2)}) \not\subset \Gamma_K(p)^{(v_0)}$ and $(\Gamma_K(p)^{(v_1)})^p \not\subset \Gamma_K(p)^{(v_0)}$ and in some sense the groups $\Gamma_K(p)/\Gamma_K(p)^{(v_0)}$ have no "simple" relations [5].

The significance of study of ramification filtration was very well understood long ago, e.g. cf. Shafarevich's Introduction to [17]. (The author also had interesting discussions on this subject in the IAS with A.Weil, P.Deligne and F.Pop.) As a matter of fact, the knowledge of ramification filtration is equivalent to the knowledge of the original field K due to the following local analogue of the Grothendieck conjecture.

Theorem 1.1. *The functor* $(K, K_{sep}) \mapsto (\Gamma_K(p), \{\Gamma_K(p)^{(v)}\}_{v\geqslant 0})$ *from* LF *to the category of profinite* p*-groups with filtration is fully faithful.*

This result was first proved in the mixed characteristic case in the context of the whole Galois group Γ_K by Mochizuki [21] as a spectacular application of p-adic Hodge-Tate theory. The case of arbitrary characteristic was established by the author by a different method in [7] under the assumption $p \neq 2$. Note that the characteristic p case was obtained via the explicit description of ramification filtration modulo the subgroup of third commutators from [2]. Then the mixed characteristic case was deduced from it via the Fontaine-Wintenberger field-of-norms functor. In paper [11] we removed the restriction $p \neq 2$ and reproved the statement in the context of the pro-p-group $\Gamma_K(p)$.

The study of ramification filtration in full generality seems not to be a realistically stated problem: it is not clear how to specify subgroups of a given profinite p-group. If we replace $\Gamma_K(p)$ by its maximal abelian quotient $\Gamma_K(p)^{ab}$ then the appropriate ramification filtration is very well known but reflects very weak information about the original filtration of $\Gamma_K(p)$. This can be seen from class field theory where we have the reciprocity map $K^* \longrightarrow \Gamma_K^{ab}$ and the ramification subgroups appear as the images of the subgroups of principal units of K^*. In particular, we can observe only integral breaks of our filtration.

As a matter of fact, the ramification subgroups can be described on the abelian level without class field theory. The reason is that cyclic extensions of K can be studied via much more elementary tools: we can use the Witt-Artin-Schreier theory in the characteristic p case and the Kummer theory in the mixed characteristic case. Trying to develop this approach to the case of nilpotent Galois groups we developed in [1, 2] a nilpotent analogue of the Witt-Artin-Schreier theory. This theory allows us to describe quite efficiently p-extensions of fields of characteristic p with Galois p-groups of nilpotent class $< p$. Such groups arise from Lie algebras due to the classical equivalence of the categories of p-groups and Lie $\mathbb{F}_p$-algebras of nilpotent class $< p$, [20]. In [1, 2, 4] we applied our theory to local fields $\mathcal{K} = k((t_0))$, where $k \simeq \mathbb{F}_{p^{N_0}}$, and constructed explicitly the sets of generators of the appropriate ramification subgroups. This result demonstrates the advantage of our techniques: it is stated in terms of extensions of scalars of involved Lie algebras but this operation does not exist in group theory.

A generalization of our approach to local fields K of mixed characteristic was sketched earlier by the author in [6]. This approach allowed us to work with the groups $\Gamma_K / \Gamma_K^{p^M} C_p(\Gamma_K)$ under the assumption that a primitive p^M-th root of unity $\zeta_{p^M} \in K$. At that time we obtained explicit constructions of our theory only modulo subgroup of third commutators. Recently, we can treat the general case. First results are related to the case $M = 1$ and can be found in [11] (we discuss them also in Subsection 1.3.6 of this paper). The case of arbitrary M as well as the case of higher dimensional local fields will be considered in upcoming papers. In the case of local fields it would be very interesting to relate our theory to constructions of "nilpotent class field theory" from [19].

Note that the main constructions of the nilpotent Artin-Schreier theory do not suggest that the basic field is local. They can be applied also to global fields but it is not clear what sort of applications we can expect in this direction.

On the other hand, we can't expect the existence of an easy "nilpotent Kummer theory" for global fields. According to anabelian philosophy, for global fields E, the quotient of $\Gamma_E(p)$ by the subgroup of third commutators should already reflect all basic properties of the field E.

1.1 Nilpotent Artin-Schreier theory

In this section we discuss basic constructions of nilpotent Artin-Schreier theory. The main reference for this theory is [2]. We shall call this version contravariant and introduce also its covariant analogue, cf. Subsection 1.1.2 below. Everywhere M is a fixed natural number.

1.1.1 Lifts modulo p^M, $M \in \mathbb{N}$.

Suppose K is a field of characteristic p and K_{sep} is a separable closure of K. Let $\{x_i\}_{i\in I}$ be a p-basis for K. This means that the elements $x_i \bmod K^{*p}$, $i \in I$, form a basis of the $\mathbb{F}_p$-module K^*/K^{*p}. Note that if E is any subfield of K_{sep} containing K then $\{x_i\}_{i\in I}$ can be taken also as a p-basis for E.

Let W_M be the functor of Witt vectors of length M. For a field $K \subset E \subset K_{sep}$, define $O_M(E)$ as the subalgebra in $W_M(E)$ generated over $W_M(\sigma^{M-1}E)$ by the Teichmuller representatives $[x_i] \in W_M(K) \subset W_M(E)$ of all x_i. Then $O_M(E)$ is a lift of E modulo p^M: it is a flat $W_M(\mathbb{F}_p)$-algebra such that $O_M(E)/pO_M(E) = E$. The system of lifts $O_M(E)$ essentially depends on the original choice of a p-basis in K. If σ is the absolute Frobenius (i.e. the morphism of p-th powers) then $W_M(\sigma)$ induces a σ-linear morphism on $O_M(E)$ and we usually denote it again by σ. Note that $O_M(E)|_{\sigma=\mathrm{id}} = W_M(\mathbb{F}_p)$, if E is normal over K then the Galois group $\mathrm{Gal}(E/K)$ acts on $O_M(E)$ and the invariants of this action coincide with $O_M(K)$.

A (continuous) automorphism $\psi \in \mathrm{Aut}(E)$ generally can't be extended to $\mathrm{Aut}O_M(E)$ if ψ changes the original p-basis. But the morphism $\sigma^{M-1}\psi$ admits "almost a lift" $\sigma^{M-1}O_M(E) \longrightarrow O_M(E)$ given by the following composition

$$\sigma^{M-1}O_M(E) \subset W_M(\sigma^{M-1}E) \xrightarrow{W_M(\sigma^{M-1}\psi)} W_M(\sigma^{M-1}E) \subset O_M(E).$$

The existence of such lift allowed us to extend the modulo p methods from [1] to the modulo p^M situation in [2, 4].

1.1.2 Covariant and contravariant nilpotent Artin-Schreier theories

Suppose L is a Lie algebra over $W_M(\mathbb{F}_p)$. For $s \in \mathbb{N}$, let $C_s(L)$ be an ideal of s-th commutators in L, e.g. $C_2(L)$, resp., $C_3(L)$, is generated by the comutators $[l_1, l_2]$, resp. $[[l_1, l_2].l_3]$, where all $l_i \in L$. The algebra L has nilpotent class $< p$ if $C_p(L) = 0$.

The basic ingredient of our theory is the equivalence of the categories of p-groups of nilpotent class $< p$ and the category of Lie $\mathbb{Z}_p$-algebras of the same nilpotent class. This equivalence can be described on the level of objects killed by p^M as follows.

Suppose L is a Lie $W_M(\mathbb{F}_p)$-algebra of nilpotent class $< p$. If A is enveloping algebra for L and J is the augmentation ideal in A then there is a natural embedding of L into A/J^p (and L can be recovered as a submodule of the module of primitive elements modulo J^p in A, cf. [1] Section 1.1). The Campbell-Hausdorff formula is the map $L \times L \longrightarrow L$,

$$(l_1, l_2) \mapsto l_1 \circ l_2 = l_1 + l_2 + \frac{1}{2}[l_1, l_2] + \dots$$

such that in A mod J^p we have $\widetilde{\exp}(l_1)\widetilde{\exp}(l_2) = \widetilde{\exp}(l_1 \circ l_2)$, where $\widetilde{\exp}(x) = \sum_{0 \leqslant i < p} x^i / i!$ is the truncated exponential. The set L can be provided with the composition law $(l_1, l_2) \mapsto l_1 \circ l_2$ which gives a group structure on L. We denote this group by $G(L)$. Clearly, this group has period p^M. Then the correspondence $L \mapsto G(L)$ is the above mentioned equivalence of the categories of p-groups of period p^M and Lie $W_M(\mathbb{F}_p)$-algebras.

Here and below we shall use the notation $L_K := L \otimes_{W_M(\mathbb{F}_p)} O_M(K)$ and $L_{K_{sep}} = L \otimes_{W_M(\mathbb{F}_p)} O_M(K_{sep})$. Then Γ_K and the absolute Frobenius σ act through the second factor on $L_{K_{sep}}$, $L_{K_{sep}}|_{\sigma=\mathrm{id}} = L$ and $(L_{K_{sep}})^{\Gamma_K} = L_K$. The covariant nilpotent Artin-Schreier theory states that for any $e \in G(L_K)$, the set $F(e) = \{f \in G(L_{K_{sep}}) \mid \sigma(f) = e \circ f\}$ is not empty and the map $g \mapsto (-f) \circ g(f)$ is a group homomorphism $\pi_f(e) : \Gamma_K \longrightarrow G(L)$. The correspondence $e \mapsto \pi_f(e)$ has the following properties:

a) if $f' \in F(e)$ then $f' = f \circ c$, where $c \in G(L)$, and $\pi_f(e)$ and $\pi_{f'}(e)$ are conjugated via c;
b) for any $\pi \in \mathrm{Hom}(\Gamma_K, G(L))$, there are $e \in G(L_K)$ and $f \in F(e)$ such that $\pi_f(e) = \pi$;
c) for appropriate elements $e, e' \in G(L_K)$ and $f, f' \in G(L_{K_{sep}})$, we have $\pi_f(e) = \pi_{f'}(e')$ iff there is an $x \in G(\mathcal{L}_K)$ such that $f' = x \circ f$ and (therefore) $e' = \sigma(x) \circ e \circ (-x)$; e and e' are called R-equivalent via $x \in G(L_K)$.

According to above properties a)–c), the correspondence $e \mapsto \pi_f(e)$ establishes an identification of the set of all R-equivalent elements in $G(L_K)$ and the set of all conjugacy classes of $\mathrm{Hom}(\Gamma_K, G(L))$.

The above theory can be proved in a similar way to its contravariant version established in [2]. In the contravariant theory for any $e \in G(L_K)$, the set $\{f \in G(L_{K_{sep}}) \mid \sigma(f) = f \circ e\}$ is not empty, the correspondence $g \mapsto g(f) \circ (-f)$

establishes a group homomorphism from Γ_K^0 to $G(L_K)$, where Γ_K^0 coincides with Γ_K as a set but has the opposite group law $(g_1g_2)^0 = g_2g_1$. (Equivalently, if $a \in K_{sep}$ then $(g_1g_2)a = g_2(g_1a)$.) We have also the properties similar to above properties a)–c) but in c) there should be $f' = f \circ x$ and $e' = x \circ e \circ (-\sigma x)$.

The both (covariant and contravariant) theories admit a pro-finite version where L becomes a profinite $W_M(\mathbb{F}_p)$-Lie algebra and the set $\mathrm{Hom}(\Gamma_K, G(L))$ is the set of all continuous group morphisms.

1.1.3 Identification η_0

Suppose $\mathcal{K} = k((t_0))$ where t_0 is a fixed uniformiser in $\mathcal{K}$ and $k \simeq \mathbb{F}_{p^{N_0}}$ with $N_0 \in \mathbb{N}$. Then $\{t_0\}$ is a p-basis for $\mathcal{K}$, and we have the appropriate system of lifts $O_M(\mathcal{E})$ modulo p^M for all subfields $\mathcal{K} \subset \mathcal{E} \subset \mathcal{K}_{sep}$. In addition, fix an element $\alpha_0 \in W(k)$ such that $\mathrm{Tr}(\alpha_0) = 1$, where Tr is the trace map for the field extension $W(k) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \supset \mathbb{Q}_p$.

Let $\mathbb{Z}^+(p) = \{a \in \mathbb{N} \mid (a, p) = 1\}$ and $\mathbb{Z}^0(p) = \mathbb{Z}^+(p) \cup \{0\}$.

For $M \in \mathbb{N}$, let $\widetilde{\mathcal{L}}_M$ be a profinite free Lie $\mathbb{Z}/p^M$-algebra with the (topological) module of generators $\mathcal{K}^*/\mathcal{K}^{*p^M}$ and $\mathcal{L}_M = \widetilde{\mathcal{L}}_M/C_p(\widetilde{\mathcal{L}}_M)$. From time to time we drop the subscript M off to simplify the notation.

Let $\mathcal{L} = \mathcal{L}_M$. Then $\mathcal{L}_k := \mathcal{L} \otimes W_M(k)$ has the generators

$$\{D_0\} \cup \{D_{an} \mid a \in \mathbb{Z}^+(p), n \in \mathbb{Z}/N_0\}$$

due to the following identifications (where $t = [t_0]$ is the Teichmuller representative of t_0):

$$\mathcal{K}^*/\mathcal{K}^{*p^M} \otimes_{W_M(\mathbb{F}_p)} W_M(k) =$$
$$\mathrm{Hom}_{W_M(\mathbb{F}_p)}(O_M(\mathcal{K})/(\sigma - \mathrm{id})O_M(\mathcal{K}), W_M(k)) =$$
$$\mathrm{Hom}_{W_M(\mathbb{F}_p)}((W_M(\mathbb{F}_p)\alpha_0) \oplus_{a\in\mathbb{Z}^+(p)} (W_M(k)t^{-a}), W_M(k)) =$$
$$W_M(k)D_0 \times \prod_{\substack{a\in\mathbb{Z}^+(p)\\ n\in\mathbb{Z}\mathrm{mod}N_0}} W_M(k)D_{an}$$

Note that the first identification uses the Witt pairing, D_0 appears from $t_0 \otimes 1 \in \mathcal{K}^*/\mathcal{K}^{*p^M} \otimes W_M(k)$ and for all $a \in \mathbb{Z}^+(p)$ and $w \in W_M(k)$, $D_{an}(wt^{-a}) = \sigma^n w$.

For any $n \in \mathbb{Z}/N_0$, set $D_{0n} = t \otimes (\sigma^n\alpha_0) = (\sigma^n\alpha_0)D_0$.

Let $e_0 = \sum_{a\in\mathbb{Z}^0(p)} t^{-a}D_{a0} \in \mathcal{L}_K$, choose $f_0 \in F(e_0)$ and set $\eta_0 = \pi_{f_0}(e_0)$. Then η_0 is a surjective homomorphism from $\Gamma_{\mathcal{K}}$ to $G(\mathcal{L})$ and it induces a group isomorphism $\Gamma_{\mathcal{K}}/\Gamma_{\mathcal{K}}^{p^M}C_p(\Gamma_{\mathcal{K}}) \simeq G(\mathcal{L})$. Note that the construction of

η_0 depends up to conjugacy only on the original choice of the uniformizer t_0 and the element $\alpha_0 \bmod p^M \in W_M(k)$. On the level of maximal abelian quotients of period p^M, η_0 induces the isomorphism of local class field theory $\Gamma_{\mathcal{K}}^{ab} \otimes_{\mathbb{Z}_p} W_M(\mathbb{F}_p) \simeq \mathcal{K}^*/\mathcal{K}^{*p^M}$.

1.1.4 Why Campbell-Hausdorff?

In this subsection it will be explained that in our theory, we are, essentially, forced to use the Campbell-Hausdorff composition law.

Assume for simplicity, that $M = 1$ and $\mathcal{K} = \mathbb{F}_p((t_0))$. Let $\mathcal{K}(p)$ be the maximal p-extension of $\mathcal{K}$ and $\Gamma_{\mathcal{K}}(p) = \mathrm{Gal}(\mathcal{K}(p)/\mathcal{K})$. For $s \in N$ and $a_1, a_2, \ldots, a_s, \ldots \in \mathbb{Z}^0(p)$, consider the elements $T_{a_1 \ldots a_s} \in \mathcal{K}(p)$ such that:

$$
\begin{aligned}
&T_{a_1}^p - T_{a_1} = t_0^{-a_1},\\
&T_{a_1 a_2}^p - T_{a_1 a_2} = t_0^{-a_1} T_{a_2}\\
&\qquad \cdots\cdots\cdots\\
&T_{a_1 \ldots a_s}^p - T_{a_1 \ldots a_s} = t_0^{-a_1} T_{a_2 \ldots a_s}\\
&\qquad \cdots\cdots\cdots
\end{aligned}
$$

Then the system $\{T_{a_1 \ldots a_s} \mid s \geqslant 0, a_i \in \mathbb{Z}^0(p)\}$ is linearly independent over $\mathcal{K}$ and if $\mathcal{M} = \bigoplus_{\substack{a_1,\ldots,a_s \\ s \geqslant 0}} \mathbb{F}_p T_{a_1 \ldots a_s}$ then $\mathcal{K}(p) = \mathcal{M} \otimes_{\mathbb{F}_p} \mathcal{K}$ and $\Gamma_{\mathcal{K}}(p)$ acts on $\mathcal{M}$ via a natural embedding $\Gamma_{\mathcal{K}}(p) \hookrightarrow \mathrm{GL}_{\mathbb{F}_p}(\mathcal{M})$. This construction would have given us an efficient approach to an explicit construction of the maximal p-extension $\mathcal{K}(p)$ if we could describe explicitly the image of $\Gamma_{\mathcal{K}}(p)$ in $\mathrm{GL}_{\mathbb{F}_p}(\mathcal{M})$.

Analyze the situation at different levels $s \geqslant 1$.

- **1st level.** Here all equations are independent and we can introduce a minimal system of generators τ_a, $a \in \mathbb{Z}^0(p)$, of $\Gamma_{\mathcal{K}}(p)$ with their explicit action via $\tau_a : T_{a_1} \mapsto T_{a_1} + \delta(a, a_1)$ at this level. (Here and below δ is the Kronecker symbol.)
- **2nd level.** Here the roots $T_{a_1 a_2}$ are not (algebraically) independent. For example, the following identity

  $$(T_{a_1} T_{a_2})^p = (T_{a_1} + t_0^{-a_1})(T_{a_2} + t_0^{-a_2}) = T_{a_1} T_{a_2} + t_0^{-a_1} T_{a_2} + t_0^{-a_2} T_{a_1} + t_0^{-(a_1+a_2)}$$

  implies under the assumption $(a_1 + a_2, p) = 1$ (and after a suitable choice of involved roots of Artin-Schreier equations) that

  $$T_{a_1} T_{a_2} = T_{a_1 a_2} + T_{a_2 a_1} + T_{a_1 + a_2}.$$

The presence of the term $T_{a_1+a_2}$ creates a problem: $\tau_{a_1+a_2}$ should act non-trivially on either $T_{a_1a_2}$ or $T_{a_2a_1}$ but they both do not depend on the index a_1+a_2. The situation can be resolved by a slight correction of involved equations. Namely, let $T_{a_1a_2}$ be such that

$$T^p_{a_1a_2} - T_{a_1a_2} = t_0^{-a_1}T_{a_2} + \eta(a_1,a_2)t_0^{-(a_1+a_2)}$$

where the constants $\eta(a_1,a_2) \in k$, $a_1, a_2 \in \mathbb{Z}^0(p)$, satisfy the relations

$$\eta(a_1,a_2) + \eta(a_2,a_1) = 1. \tag{1.1}$$

With the above correction, the elements $T_{a_1a_2}$, $a_1, a_2 \in \mathbb{Z}^0(p)$, can be chosen in such a way that we have the following:

— relations: $T_{a_1}T_{a_2} = T_{a_1a_2} + T_{a_2a_1}$;
— Galois action: $\tau_a(T_{a_1a_2}) = T_{a_1a_2} + T_{a_1}\delta(a_2,a) + \eta(a_1,a_2)\delta(a_1,a_2,a)$.

Relation (1.1) will look more natural if we introduce the constants on the first level via $\eta(a) = 1$, $a \in \mathbb{Z}^0(p)$. Then (1.1) can be rewritten as $\eta(a_1)\eta(a_2) = \eta(a_1,a_2) + \eta(a_2,a_1)$. These relations can be satisfied only if $p \neq 2$ and the simplest choice is $\eta(a_1,a_2) = 1/2$ for all a_1, a_2.

The above picture can be generalized to higher levels as follows.

- **s-th level, $s < p$.** Here we have:
  - the equations: $T^p_{a_1\dots a_s} = T_{a_1\dots a_s} + \eta(a_1)t^{-a_1}T_{a_2\dots a_s} + \dots$
    $+\eta(a_1,\dots,a_{s-1})t_0^{-(a_1+\dots+a_{s-1})}T_{a_s} + \eta(a_1,\dots,a_s)t_0^{-(a_1+\dots+a_s)}$
  - the relations: $T_{a_1\dots a_k}T_{b_1\dots b_l} = \sum T_{\text{insertions of } a\text{'s into } b\text{'s}}$, where $k+l < p$;
  - the Galois action: $\tau_a(T_{a_1\dots a_s}) = T_{a_1\dots a_s} + T_{a_1\dots a_{s-1}}\delta(a,a_s)\eta(a_s) +$
    $\dots + T_{a_1}\delta(a,a_2,\dots,a_s)\eta(a_2,\dots,a_s) + \delta(a,a_1,\dots,a_s)\eta(a_1,\dots,a_s)$
  - the constants: if $k+l < p$ then
    $\eta(a_1,\dots,a_k)\eta(b_1,\dots,b_l) = \sum \eta(\text{insertions of } a\text{'s into } b\text{'s})$
    with their simplest choice $\eta(a_1,\dots,a_s) = 1/s!$

Remark. An insertion of the ordered collection $a_1,\dots,a_k$ into the ordered collection $b_1,\dots,b_l$ is the ordered collection $c_1,\dots,c_{k+l}$ such that

— $\{1,\dots,k+l\} = \{i_1,\dots,i_k\} \coprod \{j_1,\dots,j_l\}$;
— $i_1 < \dots < i_k$ and $j_1 < \dots < j_l$;
— $a_1 = c_{i_1},\dots, a_k = c_{i_k}$ and $b_1 = c_{j_1},\dots, b_l = c_{j_l}$.

The following formalism allows us to present the above information on all levels $1 \leqslant s < p$ in the following compact way.

Let $\widetilde{\mathcal{A}}$ be a pro-finite associative $\mathbb{F}_p$-algebra with the set of free generators $\{D_a \mid a \in \mathbb{Z}^0(p)\}$. Introduce the elements of the appropriate extensions of scalars of $\mathcal{A}$

$$E = 1 + \sum_{\substack{1 \leqslant s < p \\ a_i \in \mathbb{Z}^0(p)}} \eta(a_1, \dots, a_s) t_0^{-(a_1 + \dots + a_s)} D_{a_1} \dots D_{a_s}$$

$$= \widetilde{\exp}\Big(\sum_{a \in \mathbb{Z}^0(p)} t_0^{-a} D_a \Big) \in \mathcal{A}_{\mathcal{K}},$$

$$\mathcal{F} = 1 + \sum_{\substack{1 \leqslant s < p \\ a_i \in \mathbb{Z}^0(p)}} \eta(a_1, \dots, a_s) T_{a_1 \dots a_s} D_{a_1} \dots D_{a_s} \in \mathcal{A}_{\mathcal{K}_{sep}}$$

Define the diagonal map as the morphism of $\mathbb{F}_p$-algebras

$$\Delta : \mathcal{A} \bmod \deg\ p \longrightarrow \mathcal{A} \otimes \mathcal{A} \bmod\ \deg\ p$$

such that for any $a \in \mathbb{Z}^0(p)$, $D_a \mapsto D_a \otimes 1 + 1 \otimes D_a$. Then we have the following properties:

— $\Delta(E) \equiv E \otimes E \bmod \deg p$; $\Delta(\mathcal{F}) \equiv \mathcal{F} \otimes \mathcal{F} \bmod \deg p$;
— $\sigma(\mathcal{F}) \equiv E\mathcal{F} \bmod \deg p$; $\tau_a(\mathcal{F}) \equiv \mathcal{F}\, \widetilde{\exp}(D_a) \bmod \deg p$.

Now we can verify the existence of $f \in \mathcal{L}_{\mathcal{K}_{sep}}$ such that $\mathcal{F} = \widetilde{\exp}(f)$ modulo deg p, and recover the basic relations $\sigma(f) = (\sum_a t_0^{-a} D_a) \circ f$ and $\tau_a(f) = f \circ D_a$, $a \in \mathbb{Z}^0(p)$, of our nilpotent Artin-Schreier theory.

1.2 Ramification filtration in $\mathcal{L} = \mathcal{L}_{M+1}$

In this section we describe and illustrate the main trick used in papers [1, 2, 4]. This trick allowed us to find explicit generators of ramification subgroups under the identification η_0 from Subsection 1.1.3. Remind that we work over $\mathcal{K} = k((t_0))$, where $k \simeq \mathbb{F}_{p^{N_0}}$, $N_0 \in \mathbb{N}$.

1.2.1 Auxiliary field $\mathcal{K}' = \mathcal{K}(r^*, N)$, [1, 2, 4]

The field $\mathcal{K}'$ is a totally ramified extension of $\mathcal{K}$ in $\mathcal{K}_{sep}$. It depends on two parameters: $r^* \in \mathbb{Q}$ such that $r^* > 0$ and $v_p(r^*) = 0$, and $N \in \mathbb{N}$ such that if $q = p^N$ then $b^* := r^*(q-1) \in \mathbb{N}$. Note that for a given r^*, there are infinitely many ways to choose N, in particular, we can always assume that N is sufficiently large.

By definition, $[\mathcal{K}' : \mathcal{K}] = q$ and the Herbrand function $\varphi_{\mathcal{K}'/\mathcal{K}}$ has only one edge point (r^*, r^*). It can be proved that $\mathcal{K}' = k((t_0'))$, where $t_0 = t_0'^q E(-1, t_0'^{b^*})$. Here for $w \in W(k)$,

$$E(w, X) = \exp(wX + \sigma(w)X^p/p + \cdots + \sigma^n(w)X^{p^n}/p^n + \dots) \in \mathbb{Z}_p[[X]]$$

is the Shafarevich version of the Artin-Hasse exponential.

Note that if $r^* \notin \mathbb{N}$, $\mathcal{K}'/\mathcal{K}$ is neither Galois nor a p-extension.

1.2.2 The criterion

Consider the following lifts modulo p^{M+1} with respect to the p-basis $\{t_0\}$ of $\mathcal{K}$

$$O_{M+1}(\mathcal{K}) = W_{M+1}(k)((t)) = W_{M+1}(\sigma^M\mathcal{K})[t]$$
$$O_{M+1}(\mathcal{K}_{sep}) = W_{M+1}(\sigma^M\mathcal{K}_{sep})[t] \subset W_{M+1}(\mathcal{K}_{sep}).$$

Remember that $t = [t_0] \in O_{M+1}(\mathcal{K})$ is the Teichmüller representative of t_0 in $W_{M+1}(\mathcal{K})$.

For $\mathcal{K}' = \mathcal{K}(r^*, N)$ and its uniformiser t_0' from Subsection 1.2.1 consider the appropriate lifts $O'_{M+1}(\mathcal{K}')$ and $O'_{M+1}(\mathcal{K}'_{sep})$. If $t' = [t_0']$ then t and t' can be related one-to-another in $W_{M+1}(\mathcal{K}')$ via

$$t^{p^M} = t'^{p^M q}\exp(-p^M t'^{b^*} - \cdots - pt'^{p^{M-1}b^*})E(-1, t'^{p^M b^*}).$$

This implies the following relations between the lifts for $\mathcal{K}$ and $\mathcal{K}'$

$$\sigma^M O_{M+1}(\mathcal{K}) \subset W_{M+1}(\sigma^M\mathcal{K}) \subset O'_{M+1}(\mathcal{K}')$$
$$\sigma^M O_{M+1}(\mathcal{K}_{sep}) \subset W_{M+1}(\sigma^M\mathcal{K}_{sep}) \subset O'_{M+1}(\mathcal{K}'_{sep})$$

As earlier, take $e_0 = \sum_{a\in\mathbb{Z}^0(p)} t^{-a}D_{a0} \in \mathcal{L}_{\mathcal{K}}$, $f_0 \in \mathcal{L}_{\mathcal{K}_{sep}}$ such that $\sigma f_0 = e_0 \circ f_0$ and consider $\pi_{f_0}(e_0) : \Gamma_{\mathcal{K}} \longrightarrow G(\mathcal{L})$. Similarly, let $e_0' = \sum_{a\in\mathbb{Z}^0(p)} t'^{-a}D_{a,-N}$, choose $f_0' \in \mathcal{L}_{\mathcal{K}_{sep}}$ such that $\sigma f_0' = e_0' \circ f_0'$ and consider $\pi_{f_0'}(e_0') : \Gamma_{\mathcal{K}'} \longrightarrow G(\mathcal{L})$.

For $Y \in \mathcal{L}_{\mathcal{K}_{sep}}$ and an ideal $\mathcal{I}$ in $\mathcal{L}$, define the field of definition of $Y \bmod \mathcal{I}_{\mathcal{K}_{sep}}$ over $\mathcal{K}$ as $\mathcal{K}(Y \bmod \mathcal{I}_{\mathcal{K}_{sep}}) := \mathcal{K}_{sep}^{\mathcal{H}}$, where $\mathcal{H} = \{g \in \Gamma_{\mathcal{K}} \mid g(Y) \equiv Y \bmod \mathcal{I}_{\mathcal{K}_{sep}}\}$.

For any finite field extension $\mathcal{E}/\mathcal{K}$ in $\mathcal{K}_{sep}$ define its biggest ramification number $v(\mathcal{E}/\mathcal{K}) = \max\{v \mid \Gamma_{\mathcal{K}}^{(v)} \text{acts non-trivially on } \mathcal{E}\}$.

For $v_0 \in \mathbb{Q}_{>0}$, let the ideal $\mathcal{L}^{(v_0)}$ of $\mathcal{L}$ be such that $G(\mathcal{L}^{(v_0)}) = \eta_0(\Gamma_{\mathcal{K}}^{(v_0)})$. Let $f_M = \sigma^M f_0$ and $f_M' = \sigma^M f_0'$. Our method from [1, 2, 4] is based on the following criterion.