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Ramsey classes: examples and constructions

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Abstract

This article is concerned with classes of relational structures that are closed under taking substructures and isomorphism, that have the joint embedding property, and that furthermore have the *Ramsey property*, a strong combinatorial property which resembles the statement of Ramsey's classic theorem. Such classes of structures have been called *Ramsey classes*. Nešetřil and Rödl showed that they have the *amalgamation property*, and therefore each such class has a homogeneous Fraïssé limit. Ramsey classes have recently attracted attention due to a surprising link with the notion of extreme amenability from topological dynamics. Other applications of Ramsey classes include reduct classification of homogeneous structures.

We give a survey of the various fundamental Ramsey classes and their (often tricky) combinatorial proofs, and about various methods to derive new Ramsey classes from known Ramsey classes. Finally, we state open problems related to a potential classification of Ramsey classes.

Contents

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1 Introduction

Let $\mathcal{C}$ be a class of finite relational structures. Then $\mathcal{C}$ has the *Ramsey property* if it satisfies a property that resembles the statement of Ramsey's theorem: for all $\mathfrak{A}, \mathfrak{B} \in \mathcal{C}$ there exists $\mathfrak{C} \in \mathcal{C}$ such that for every colouring of the embeddings of $\mathfrak{A}$ into $\mathfrak{C}$ with finitely many colours there exists a 'monochromatic copy' of $\mathfrak{B}$ in $\mathfrak{C}$, that is, an embedding e of $\mathfrak{B}$ into $\mathfrak{C}$ such that all embeddings of $\mathfrak{A}$ into the image of e have the same colour. An example of a class of structures with the Ramsey property is the class of all finite linearly ordered sets; this is Ramsey's theorem [45]. Another example of a class with the Ramsey property is the class of all ordered finite graphs, that is, structures $(V; E, \preceq)$ where V is a finite set, E the undirected edge relation, and $\preceq$ a linear order on V; this result has been discovered by Nešetřil and Rödl [41], and, independently, Abramson and Harrington [1].

In this article we will be concerned exclusively with classes $\mathcal{C}$ that are closed under taking substructures and isomorphism, and that have the *joint embedding property*: whenever $\mathfrak{A}, \mathfrak{B} \in C$, then there exists a $\mathfrak{C} \in \mathcal{C}$ such that both $\mathfrak{A}$ and $\mathfrak{B}$ embed into $\mathfrak{C}$. These are precisely the classes $\mathcal{C}$ for which there exists a countably infinite structure Γ such that a structure belongs to $\mathcal{C}$ if and only if it embeds into Γ. In this case, following Fraïssé's terminology, we say that $\mathcal{C}$ is the *age* of Γ. Our structures always have an at most countable signature. A class $\mathcal{C}$ will be called a *Ramsey class* [38] if it has the Ramsey property, is closed under isomorphisms and substructures, and has the joint embedding property. It is an open research problem, raised in [38], whether Ramsey classes can be *classified* in some sense that needs to be specified.

It has been shown by Nešetřil [38] that Ramsey classes have the *amalgamation property*, a central property in model theory. A class of structures $\mathcal{C}$ has the amalgamation property if for all $\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{C}$ with embeddings e_i of $\mathfrak{A}$ into $\mathfrak{B}_i$, for $i \in \{1, 2\}$, there exist $\mathfrak{C} \in \mathcal{C}$ and embeddings f_i of $\mathfrak{B}_i$ into $\mathfrak{C}$ such that $f_1(e_1(a)) = f_2(e_2(a))$ for all elements a of $\mathfrak{A}$. A class of finite structures $\mathcal{C}$ is an *amalgamation class* if it is closed under induced substructures, isomorphism, and has the amalgamation property. By Fraïssé's theorem (which will be recalled in Section 2.5) for every amalgamation class $\mathcal{C}$ there exists a countably infinite structure Γ of age $\mathcal{C}$ which is *homogeneous*, that is, any isomorphism between finite substructures of Γ can be extended to an automorphism of Γ. The structure Γ is in fact unique up to isomorphism, and called the *Fraïssé limit of* $\mathcal{C}$. In our example above where $\mathcal{C}$ is the class of all finite linearly ordered sets $(V; <)$, the Fraïssé limit is isomorphic to $(\mathbb{Q}; <)$, that is, the linear order of the rationals.

The age of a homogeneous structure with a finite relational signature is in general not Ramsey. However, quite surprisingly, homogeneous structures with finite relational signature typically have a homogeneous *expansion* by finitely many relations such that the age of the resulting structure is Ramsey. The question whether we can replace in the previous sentence the word 'typically' by the word 'always' appeared in discussions of the author with Michael Pinsker and Todor Tsankov in 2010, and has been asked, first implicitly in a conference publication [11], then explicitly in the journal version. The question motivates much of the material present in this article, so we prominently state it here as follows.

Conjecture 1.1 (Ramsey expansion conjecture) *Let Γ be a homogeneous structure with finite relational signature. Then Γ has a homogeneous expansion by finitely many relations whose age has the Ramsey property.*

This conjecture has explicitly been confirmed for all countable homogeneous directed graphs in [31] (those graphs have been classified by Cherlin [17]), and other homogeneous structures of interest [26]. The Ramsey expansion conjecture has several variants that are formally unrelated, but related in spirit; we will come back to this in the final section of the article. There we also discuss that the conjecture can be translated into questions in topological dynamics which are of independent interest.

This text has its focus on the combinatorial aspects of the theory, rather than the links with topological dynamics. What we do find convenient, though, is the usage of concepts from model theory to present the results: instead of manipulating classes of structures $\mathcal{C}$ that are closed under substructures, isomorphism, and that have the joint embedding and the amalgamation property, it is often more convenient to directly manipulate homogeneous structures of age $\mathcal{C}$.

Outline of the article. In Section 2.2 we give a self-contained introduction to the basics of Ramsey classes, including the proofs of some well-known and easy observations about them. In Section 3 we show how to derive new Ramsey classes from known ones; this section contains various facts or proofs that have not explicitly appeared in the literature yet.

- In Section 3.3 we have basic results about the Ramsey properties of interpreted structures that have not been formulated previously in this form, but that are not difficult to show via variations of the so-called *product Ramsey theorem*.
- In Section 3.4 we present a new non-topological proof, due to Miodrag Sokic, of a known fact from [11] about expanding Ramsey classes with constants.
- In Section 3.5 and 3.6 we present generalisations of results from [6] about the Ramsey properties of model-companions and model-complete cores of ω-categorical structures.

Some fundamental Ramsey classes cannot be constructed by the general construction principles from Section 3. The most powerful tool that we have to prove Ramsey theorems from scratch is the *partite method*, developed in the 70s and 80s, most notably by Nešetřil and Rödl, which we present in Section 4. With this method we will show that the following classes are Ramsey: the class of all ordered graphs, the class of all ordered triangle-free graphs, or more generally the class of all ordered structures given by a set of homomorphically forbidden irreducible substructures.

There are also Ramsey classes with finite relational signature where it is not clear how to show the Ramsey property with the partite method, to the best of my knowledge. We will see such an example, based on Ramsey theorems for tree-like structures, in Section 5.

When we want to make progress on Conjecture 1.1, we need a better understanding of the type of expansion needed to turn a homogeneous structure in a finite language into a Ramsey structure. Very often, this can be done by adding a linear ordering to the signature (a partial explanation for this is given in Section 2.8). But not any linear ordering might do the job; a crucial property for finding the right ordering is the so-called *ordering property*, which is a classical notion in structural Ramsey theory. We will present in Section 6 a powerful condition that implies that a Ramsey class has the ordering property with respect to some given ordering.

Finally, in Section 7, we discuss the mentioned link between Ramsey theory and topological dynamics, then present an application of Ramsey theory for classifying reducts of homogeneous structures, and conclude with some open problems related to Conjecture 1.1.

2 Ramsey classes: definition, examples, background

The definition of Ramsey classes is inspired by the statement of the classic theorem of Ramsey, which we therefore recall in the next subsection, before defining the Ramsey property in Section 2.2 and Ramsey classes in Section 2.3.

There are two important necessary conditions for a class to be Ramsey: *rigidity* (Section 2.4) and *amalgamation* (Section 2.5). We will see examples that show that these two conditions are not sufficient (Section 2.6). The Ramsey property of a Ramsey class $\mathcal{C}$ can be seen as a property of the automorphism group of the Fraïssé limit of $\mathcal{C}$; this perspective is discussed in Sections 2.7 and 2.8.

2.1 Ramsey's theorem

The set of positive integers is denoted by $\mathbb{N}$, and the set $\{1,\dots,n\}$ is denoted by $[n]$. For $M, S \subseteq \mathbb{N}$ we write $\binom{M}{S}$ for the set of all order-preserving maps from S into M. When f is a map, and $\mathcal{S}$ is a set of maps whose range equals the domain of f, then $f \circ \mathcal{S}$ denotes the set $\{f \circ e \mid e \in \mathcal{S}\}$. A proof of Ramsey's theorem can be found in almost any textbook on combinatorics.

Theorem 2.1 (Ramsey's theorem [45]**)** *For all $r, m, k \in \mathbb{N}$ there is a positive integer g such that for every $\chi \colon \binom{[g]}{[k]} \to [r]$ there exists an $f \in \binom{[g]}{[m]}$ such that $|\chi(f \circ \binom{[m]}{[k]})| \leq 1$.*

2.2 The Ramsey property

In this section we define the Ramsey property for classes of structures. All structures in this article have an at most countable domain, and have an at most countable signature. Typically, the signature will be relational and even finite; but many results generalise to signatures that are infinite

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and also contain function symbols. In Section 3.4 it will be useful to consider signatures that also contain constant symbols (i.e., function symbols of arity zero).

Let τ be a relational signature, let $\mathfrak{B}$ be a τ-structure. For $R \in \tau$, we write $R^{\mathfrak{B}}$ for the corresponding relation of $\mathfrak{B}$. Typically, the domain of $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ will be denoted by A, B, C, respectively. Let A be a subset of the domain B of $\mathfrak{B}$. Then the *substructure* of $\mathfrak{B}$ *induced by* A is the τ-structure $\mathfrak{A}$ with domain A such that for every relation symbol $R \in \tau$ of arity k we have $R^{\mathfrak{A}} = R^{\mathfrak{B}} \cap A^k$.

If τ is not a purely relational signature, but also contains constant symbols, then every substructure $\mathfrak{A}$ of $\mathfrak{B}$ must contain for every constant symbol c in τ the element $c^{\mathfrak{B}}$, and $c^{\mathfrak{A}} = c^{\mathfrak{B}}$. An *embedding* of $\mathfrak{B}$ into $\mathfrak{A}$ is a mapping f from B to A which is an isomorphism between $\mathfrak{B}$ and the substructure induced by the image of f in $\mathfrak{B}$. This substructure will also be called a *copy* of $\mathfrak{A}$ in $\mathfrak{B}$. We write $\binom{\mathfrak{B}}{\mathfrak{A}}$ for the set of all embeddings of $\mathfrak{A}$ into $\mathfrak{B}$.

Definition 2.2 (The partition arrow) When $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ are τ-structures, and $r \in \mathbb{N}$, then we write $\mathfrak{C} \to (\mathfrak{B})^{\mathfrak{A}}_r$ if for all $\chi \colon \binom{\mathfrak{C}}{\mathfrak{A}} \to [r]$ there exists an $f \in \binom{\mathfrak{C}}{\mathfrak{B}}$ such that $|\chi(f \circ \binom{\mathfrak{B}}{\mathfrak{A}})| \leq 1$.

We would like to mention that in some papers, the partition arrow is defined for the situation where $\binom{\mathfrak{B}}{\mathfrak{A}}$ does not denote the set of embeddings of $\mathfrak{A}$ into $\mathfrak{B}$, but the set of copies of $\mathfrak{A}$ in $\mathfrak{B}$. These two definitions are closely related; the article [36] is specifically about this difference. Also [27] and [55] treat the relationship between the two definitions.

In analogy to the statement of Ramsey's theorem, we can now define the Ramsey property for a class of relational structures.

Definition 2.3 (The Ramsey property) A class $\mathcal{C}$ of finite structures has the *Ramsey property* if for all $\mathfrak{A}, \mathfrak{B} \in \mathcal{C}$ and $k \in \mathbb{N}$ there exists a $\mathfrak{C} \in \mathcal{C}$ such that $\mathfrak{C} \to (\mathfrak{B})^{\mathfrak{A}}_k$.

Example 2.4 The class of all finite linear orders, denoted by $\mathcal{LO}$, has the Ramsey property. This is a reformulation of Theorem 2.1.

The following well-known fact shows that we can always work with 2-colourings instead of general colourings when we want to prove that a certain class has the Ramsey property.

Lemma 2.5 *Let $\mathcal{C}$ be a class of structures, and $\mathfrak{A} \in \mathcal{C}$. Then for every $\mathfrak{B} \in \mathcal{C}$ and $r \in \mathbb{N}$ there exists a $\mathfrak{C} \in \mathcal{C}$ such that $\mathfrak{C} \to (\mathfrak{B})^{\mathfrak{A}}_r$ if and only if for every $\mathfrak{B} \in \mathcal{C}$ there exists a $\mathfrak{C} \in \mathcal{C}$ such that $\mathfrak{C} \to (\mathfrak{B})^{\mathfrak{A}}_2$.*

Proof Suppose that for every $\mathfrak{B} \in \mathcal{C}$ there exists a $\mathfrak{C} \in \mathcal{C}$ such that $\mathfrak{C} \to (\mathfrak{B})^{\mathfrak{A}}_2$. We inductively define a sequence $\mathfrak{C}_1, \ldots, \mathfrak{C}_{r-1}$ of structures in $\mathcal{C}$ as follows. Let $\mathfrak{C}_1$ be such that $\mathfrak{C}_1 \to (\mathfrak{B})^{\mathfrak{A}}_2$. For $i \in \{2, \ldots, r-1\}$, let $\mathfrak{C}_i$ be such that $\mathfrak{C}_i \to (\mathfrak{C}_{i-1})^{\mathfrak{A}}_2$. We leave it to the reader to verify that $\mathfrak{C}_{r-1} \to (\mathfrak{B})^{\mathfrak{A}}_r$. □

2.3 The joint embedding property and Ramsey classes

We say that a class of structures $\mathcal{C}$ is *closed under substructures* if for every $\mathfrak{B} \in \mathcal{C}$, all substructures of $\mathfrak{B}$ are also in $\mathcal{C}$. The class $\mathcal{C}$ is *closed under isomorphism* if for every $\mathfrak{B} \in \mathcal{C}$, all structures that are isomorphic to $\mathfrak{B}$ are also in $\mathcal{C}$. In this article, we will focus on classes of finite structures that are closed under induced substructures and isomorphism, and that have the joint embedding property. Recall from the introduction that $\mathcal{C}$ has the joint embedding property if for every $\mathfrak{A}, \mathfrak{B} \in C$, there exists a $\mathfrak{C} \in \mathcal{C}$ such that both $\mathfrak{A}$ and $\mathfrak{B}$ embed into $\mathfrak{C}$. Such classes of structures naturally arise as follows; see e.g. [25].

Proposition 2.6 *A class of finite structures $\mathcal{C}$ is closed under substructures, isomorphism, and has the joint embedding property if and only if there exists a countable structure Γ whose age equals $\mathcal{C}$.*

Proposition 2.6 is the main motivation why we exclusively work with classes of structures that are closed under substructures; however, as demonstrated in a recent paper by Zucker [55], several Ramsey results and techniques can meaningfully be extended to isomorphism-closed classes that only satisfy the joint embedding property and amalgamation, but that are not necessarily closed under substructures.

Definition 2.7 (Ramsey class) Let τ be an at most countable relational signature. A class of finite τ-structures is called a *Ramsey class* if it is closed under substructures, isomorphism, and has the joint embedding and the Ramsey property.

Examples of Ramsey classes will be presented below, in Example 2.12, or more generally, in Example 2.13. The following can be shown by a simple compactness argument.

Proposition 2.8 *Let Γ be a structure of age $\mathcal{C}$. Then $\mathcal{C}$ is a Ramsey class if and only if for all $\mathfrak{A}, \mathfrak{B} \in \mathcal{C}$ and $r \in \mathbb{N}$ we have that $\Gamma \to (\mathfrak{B})^{\mathfrak{A}}_r$.*

Proof Let $\mathfrak{A}, \mathfrak{B} \in \mathcal{C}$, and $r \in \mathbb{N}$ an integer. When k is the cardinality of $\binom{\mathfrak{B}}{\mathfrak{A}}$, then for any structure $\mathfrak{C}$ the fact that $\mathfrak{C} \to (\mathfrak{B})^{\mathfrak{A}}_r$ can equivalently be expressed in terms of r-colourability of a certain k-uniform hypergraph, defined as follows. Let $G = (V; E)$ be the structure whose vertex set V is $\binom{\mathfrak{C}}{\mathfrak{A}}$, and where $(e_1, \ldots, e_k) \in E$ if there exists an $f \in \binom{\mathfrak{C}}{\mathfrak{B}}$ such that $f \circ \binom{\mathfrak{B}}{\mathfrak{A}} = \{e_1, \ldots, e_k\}$. Let $H = ([r]; E)$ be the structure where E contains all tuples except for the tuples $(1, \ldots, 1), \ldots, (r, \ldots, r)$. Then $\mathfrak{C} \not\to (\mathfrak{B})^{\mathfrak{A}}_r$ if and only if G does not homomorphically map to H. An easy and well-known compactness argument (see e.g. Lemma 3.1.5 in [5]) shows that this is the case if and only if some finite substructure of G does not homomorphically map to H. Thus, $\Gamma \to (\mathfrak{B})^{\mathfrak{A}}_r$ if and only if $\mathfrak{C} \to (\mathfrak{B})^{\mathfrak{A}}_r$ for all finite substructures $\mathfrak{C}$ of Γ. □

2.4 Ramsey degrees and rigidity

Let $\mathcal{C}$ be a class of structures with the Ramsey property. In this section we will see that each structure in $\mathcal{C}$ must be *rigid*, that is, it has no automorphism other than the identity.

Definition 2.9 (Ramsey degrees) Let $\mathcal{C}$ be a class of structures and let $\mathfrak{A} \in \mathcal{C}$. We say that $\mathfrak{A}$ has *Ramsey degree k (in $\mathcal{C}$)* if $k \in \mathbb{N}$ is least such that for any $\mathfrak{B} \in \mathcal{C}$ and for any $r \in \mathbb{N}$ there exists a $\mathfrak{C} \in \mathcal{C}$ such that for any r-colouring χ of $\binom{\mathfrak{C}}{\mathfrak{A}}$ there is an $f \in \binom{\mathfrak{C}}{\mathfrak{B}}$ such that $|\chi(f \circ \binom{\mathfrak{B}}{\mathfrak{A}})| \leq k$.

Hence, by definition, $\mathcal{C}$ has the Ramsey property if every $\mathfrak{A} \in \mathcal{C}$ has Ramsey degree one.

Lemma 2.10 *Let $\mathcal{C}$ be a class of finite structures. Then for every $\mathfrak{A} \in \mathcal{C}$, the Ramsey degree of $\mathfrak{A}$ in $\mathcal{C}$ is at least $|\operatorname{Aut}(\mathfrak{A})|$.*

Proof We have to show that for some $\mathfrak{B} \in \mathcal{C}$ and $r \in \mathbb{N}$, every $\mathfrak{C} \in \mathcal{C}$ can be r-coloured such that for all $f \in \binom{\mathfrak{C}}{\mathfrak{B}}$ we have $|\chi(f \circ \binom{\mathfrak{B}}{\mathfrak{A}})| \geq |\operatorname{Aut}(\mathfrak{A})|$. We choose $\mathfrak{B} := \mathfrak{A}$ and $r := |\operatorname{Aut}(\mathfrak{A})|$.

Let $\mathfrak{C} \in \mathcal{C}$ be arbitrary. Define an equivalence relation $\sim$ on $\binom{\mathfrak{C}}{\mathfrak{A}}$ by setting $f \sim g$ if there exists an $h \in \operatorname{Aut}(\mathfrak{A})$ such that $f = g \circ h$. Let $f_1, \ldots, f_t$ be a list of representatives for the equivalence classes of $\sim$. Define $\chi \colon \binom{\mathfrak{C}}{\mathfrak{A}} \to \operatorname{Aut}(\mathfrak{A})$ as follows. For $f \in \binom{\mathfrak{C}}{\mathfrak{A}}$, let i be the unique i such that $f_i \sim f$. Define $\chi(f) = h$ if $f = f_i \circ h$. Now let $e \in \binom{\mathfrak{C}}{\mathfrak{A}}$ be arbitrary. Then $|\chi(e \circ \binom{\mathfrak{A}}{\mathfrak{A}})| = |\operatorname{Aut}(\mathfrak{A})|$. □

Corollary 2.11 *Let $\mathcal{C}$ be a class with the Ramsey property. Then all $\mathfrak{A}$ in $\mathcal{C}$ are rigid.*

It follows that in particular the class of all finite graphs does *not* have the Ramsey property. Frequently, a class without the Ramsey property can be made Ramsey by expanding its members appropriately with a linear ordering (the expanded structures are clearly rigid).

Example 2.12 Abramson and Harrington [1] and independently Nešetřil and Rödl [39] showed that for any relational signature τ, the class $\mathcal{C}$ of all finite *linearly ordered* τ-structures has the Ramsey property. That is, the members of $\mathcal{C}$ are finite structures $\mathfrak{A} = (A; \preceq, R_1, R_2, \ldots)$ for some fixed signature $\tau = \{\preceq, R_1, R_2, \ldots\}$ where $\preceq$ denotes a linear order of A.

A shorter and simpler proof of this substantial result, based on the *partite method*, can be found in [40] and [37] and will be presented in Section 4.

For a class of finite τ-structures $\mathcal{N}$, we write $\mathrm{Forb}(\mathcal{N})$ for the class of all finite τ-structures that does not admit a homomorphism from any structure in $\mathcal{N}$.

Example 2.13 The classes from Example 2.12 have been further generalised by Nešetřil and Rödl [39] as follows. Suppose that $\mathcal{N}$ is a (not necessarily finite) class of structures $\mathfrak{F}$ with finite relational signature τ such that for all elements u, v of $\mathfrak{F}$ there is a tuple in a relation $R^{\mathfrak{F}}$ for $R \in \tau$ that contains both u and v. Such structures have been called *irreducible* in the Ramsey theory literature. Then the class of all expansions of the structures in $\mathcal{C} := \mathrm{Forb}(\mathcal{N})$ by a linear order has the Ramsey property. Again, there is a proof based on the partite method, which will be presented in Section 4. This is indeed a generalization since we obtain the classes from Example 2.12 by taking $\mathcal{N} = \emptyset$.

2.5 The amalgamation property

The Ramsey classes we have seen so far will look familiar to model theorists. As mentioned in the introduction, the fact that all of the above Ramsey classes could be described as the age of a homogeneous structure is not a coincidence.

Theorem 2.14 ([38]) *Let τ be a relational signature, and let $\mathcal{C}$ be a class of finite τ-structures that is closed under isomorphism, and has the joint embedding property. If $\mathcal{C}$ has the Ramsey property, then it also has the amalgamation property.*

Proof Let $\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2$ be members of $\mathcal{C}$ such that there are embeddings $e_i \in \binom{\mathfrak{B}_i}{\mathfrak{A}}$ for $i = 1$ and $i = 2$. Since $\mathcal{C}$ has the joint embedding property, there exists a structure $\mathfrak{C} \in \mathcal{C}$ with embeddings f_1, f_2 of $\mathfrak{B}_1$ and $\mathfrak{B}_2$ into

$\mathfrak{C}$. If $f_1 \circ e_1 = f_2 \circ e_2$, then $\mathfrak{C}$ shows that $\mathfrak{B}_1$ and $\mathfrak{B}_2$ amalgamate over $\mathfrak{A}$, so assume otherwise.

Let $\mathfrak{D} \in \mathcal{C}$ be such that $\mathfrak{D} \to (\mathfrak{C})^{\mathfrak{A}}_2$. Define a colouring $\chi \colon \binom{\mathfrak{D}}{\mathfrak{A}} \to [2]$ as follows. For $g \in \binom{\mathfrak{D}}{\mathfrak{A}}$, let $\chi(g) = 1$ if there is a $t \in \binom{\mathfrak{D}}{\mathfrak{C}}$ such that $g = t \circ f_1 \circ e_1$, and $\chi(g) = 0$ otherwise. Since $\mathfrak{D} \to (\mathfrak{C})^{\mathfrak{A}}_2$, there exists a $t_0 \in \binom{\mathfrak{D}}{\mathfrak{C}}$ such that $|\chi(t_0 \circ \binom{\mathfrak{C}}{\mathfrak{A}})| = 1$. Note that $\chi(t_0 \circ f_1 \circ e_1) = 1$ by the definition of χ. It follows that $\chi(t_0 \circ h) = 1$ for all $h \in \binom{\mathfrak{C}}{\mathfrak{A}}$. In particular $\chi(t_0 \circ f_2 \circ e_2) = 1$, because $f_2 \circ e_2 \in \binom{\mathfrak{C}}{\mathfrak{A}}$. Thus, by the definition of χ, there exists a $t_1 \in \binom{\mathfrak{D}}{\mathfrak{C}}$ such that $t_1 \circ f_1 \circ e_1 = t_0 \circ f_2 \circ e_2$ (here we use that the structure $\mathfrak{A}$ must be rigid, by Corollary 2.11). This shows that $\mathfrak{D}$ together with the embeddings $t_1 \circ f_1 \colon \mathfrak{B}_1 \to \mathfrak{D}$ and $t_0 \circ f_2 \colon \mathfrak{B}_2 \to \mathfrak{D}$ is an amalgam of $\mathfrak{B}_1$ and $\mathfrak{B}_2$ over $\mathfrak{A}$. □

Definition 2.15 (Amalgamation class) An isomorphism-closed class of finite structures with an at most countable relational signature that has the amalgamation property (defined in the introduction), and that is closed under taking induced substructures, is called an *amalgamation class*.

Theorem 2.16 (Fraïssé [20, 21]; see [25]) *Let τ be a countable relational signature and let $\mathcal{C}$ be an amalgamation class of τ-structures. Then there is a homogeneous and at most countable τ-structure $\mathfrak{C}$ whose age equals $\mathcal{C}$. The structure $\mathfrak{C}$ is unique up to isomorphism, and called the* Fraïssé limit *of $\mathcal{C}$.*

Example 2.17 The Fraïssé limit of the class of all finite linear orders is isomorphic to $(\mathbb{Q}; <)$, the order of the rationals. The Fraïssé limit of the class of all graphs is the so-called random graph (or Rado graph); see e.g. [15].

We also have the following converse of Theorem 2.16.

Theorem 2.18 (Fraïssé; see [25]) *Let Γ be a homogeneous relational structure. Then the age of Γ is an amalgamation class.*

As we have seen, there is a close connection between amalgamation classes and homogeneous structures, and we therefore make the following definition.

Definition 2.19 (Ramsey structure) A homogeneous structure Γ is called *Ramsey* if the age of Γ has the Ramsey property.