

Cambridge University Press

978-1-107-46249-6 - O-Minimality and Diophantine Geometry: London Mathematical Society Lecture Note Series: 421

Edited by G. O. Jones and A. J. Wilkie

Frontmatter

[More information](#)

LONDON MATHEMATICAL SOCIETY LECTURE NOTE SERIES

Managing Editor: Professor M. Reid, Mathematics Institute,
University of Warwick, Coventry CV4 7AL, United Kingdom

The titles below are available from booksellers, or from Cambridge University Press at
<http://www.cambridge.org/mathematics>

- 296 Lectures on invariant theory, I. DOLGACHEV
- 297 The homotopy category of simply connected 4-manifolds, H.-J. BAUES
- 298 Higher operads, higher categories, T. LEINSTER (ed)
- 299 Kleinian groups and hyperbolic 3-manifolds, Y. KOMORI, V. MARKOVIC & C. SERIES (eds)
- 300 Introduction to Möbius differential geometry, U. HERTRICH-JEROMIN
- 301 Stable modules and the D(2)-problem, F.E.A. JOHNSON
- 302 Discrete and continuous nonlinear Schrödinger systems, M.J. ABLOWITZ, B. PRINARI & A.D. TRUBATCH
- 303 Number theory and algebraic geometry, M. REID & A. SKOROBOGATOV (eds)
- 304 Groups St Andrews 2001 in Oxford I, C.M. CAMPBELL, E.F. ROBERTSON & G.C. SMITH (eds)
- 305 Groups St Andrews 2001 in Oxford II, C.M. CAMPBELL, E.F. ROBERTSON & G.C. SMITH (eds)
- 306 Geometric mechanics and symmetry, J. MONTALDI & T. RATIU (eds)
- 307 Surveys in combinatorics 2003, C.D. WENSLEY (ed.)
- 308 Topology, geometry and quantum field theory, U.L. TILLMANN (ed)
- 309 Corings and comodules, T. BRZEZINSKI & R. WISBAUER
- 310 Topics in dynamics and ergodic theory, S. BEZUGLYI & S. KOLYADA (eds)
- 311 Groups: topological, combinatorial and arithmetic aspects, T.W. MÜLLER (ed)
- 312 Foundations of computational mathematics, Minneapolis 2002, F. CUCKER *et al* (eds)
- 313 Transcendental aspects of algebraic cycles, S. MÜLLER-STACH & C. PETERS (eds)
- 314 Spectral generalizations of line graphs, D. CVETKOVIĆ, P. ROWLINSON & S. SIMIĆ
- 315 Structured ring spectra, A. BAKER & B. RICHTER (eds)
- 316 Linear logic in computer science, T. EHRHARD, P. RUET, J.-Y. GIRARD & P. SCOTT (eds)
- 317 Advances in elliptic curve cryptography, I.F. BLAKE, G. SEROUSSI & N.P. SMART (eds)
- 318 Perturbation of the boundary in boundary-value problems of partial differential equations, D. HENRY
- 319 Double affine Hecke algebras, I. CHEREDNIK
- 320 L-functions and Galois representations, D. BURNS, K. BUZZARD & J. NEKOVAŘ (eds)
- 321 Surveys in modern mathematics, V. PRASOLOV & Y. ILYASHENKO (eds)
- 322 Recent perspectives in random matrix theory and number theory, F. MEZZADRI & N.C. SNAITH (eds)
- 323 Poisson geometry, deformation quantisation and group representations, S. GUTT *et al* (eds)
- 324 Singularities and computer algebra, C. LOSSEN & G. PFISTER (eds)
- 325 Lectures on the Ricci flow, P. TOPPING
- 326 Modular representations of finite groups of Lie type, J.E. HUMPHREYS
- 327 Surveys in combinatorics 2005, B.S. WEBB (ed)
- 328 Fundamentals of hyperbolic manifolds, R. CANARY, D. EPSTEIN & A. MARDEN (eds)
- 329 Spaces of Kleinian groups, Y. MINSKY, M. SAKUMA & C. SERIES (eds)
- 330 Noncommutative localization in algebra and topology, A. RANICKI (ed)
- 331 Foundations of computational mathematics, Santander 2005, L.M. PARDO, A. PINKUS, E. SÜÜLI & M.J. TODD (eds)
- 332 Handbook of tilting theory, L. ANGELERI HÜGEL, D. HAPPEL & H. KRAUSE (eds)
- 333 Synthetic differential geometry (2nd Edition), A. KOCK
- 334 The Navier–Stokes equations, N. RILEY & P. DRAZIN
- 335 Lectures on the combinatorics of free probability, A. NICA & R. SPEICHER
- 336 Integral closure of ideals, rings, and modules, I. SWANSON & C. HUNEKE
- 337 Methods in Banach space theory, J.M.F. CASTILLO & W.B. JOHNSON (eds)
- 338 Surveys in geometry and number theory, N. YOUNG (ed)

Cambridge University Press

978-1-107-46249-6 - O-Minimality and Diophantine Geometry: London Mathematical Society Lecture Note Series: 421

Edited by G. O. Jones and A. J. Wilkie

Frontmatter

[More information](#)

- 339 Groups St Andrews 2005 I, C.M. CAMPBELL, M.R. QUICK, E.F. ROBERTSON & G.C. SMITH (eds)
- 340 Groups St Andrews 2005 II, C.M. CAMPBELL, M.R. QUICK, E.F. ROBERTSON & G.C. SMITH (eds)
- 341 Ranks of elliptic curves and random matrix theory, J.B. CONREY, D.W. FARMER, F. MEZZADRI & N.C. SNAITH (eds)
- 342 Elliptic cohomology, H.R. MILLER & D.C. RAVENEL (eds)
- 343 Algebraic cycles and motives I, J. NAGEL & C. PETERS (eds)
- 344 Algebraic cycles and motives II, J. NAGEL & C. PETERS (eds)
- 345 Algebraic and analytic geometry, A. NEEMAN
- 346 Surveys in combinatorics 2007, A. HILTON & J. TALBOT (eds)
- 347 Surveys in contemporary mathematics, N. YOUNG & Y. CHOI (eds)
- 348 Transcendental dynamics and complex analysis, P.J. RIPON & G.M. STALLARD (eds)
- 349 Model theory with applications to algebra and analysis I, Z. CHATZIDAKIS, D. MACPHERSON, A. PILLAY & A. WILKIE (eds)
- 350 Model theory with applications to algebra and analysis II, Z. CHATZIDAKIS, D. MACPHERSON, A. PILLAY & A. WILKIE (eds)
- 351 Finite von Neumann algebras and masas, A.M. SINCLAIR & R.R. SMITH
- 352 Number theory and polynomials, J. MCKEE & C. SMYTH (eds)
- 353 Trends in stochastic analysis, J. BLATH, P. MÖRTERS & M. SCHEUTZOW (eds)
- 354 Groups and analysis, K. TENT (ed)
- 355 Non-equilibrium statistical mechanics and turbulence, J. CARDY, G. FALKOVICH & K. GAWEDZKI
- 356 Elliptic curves and big Galois representations, D. DELBOURGO
- 357 Algebraic theory of differential equations, M.A.H. MACCALLUM & A.V. MIKHAILOV (eds)
- 358 Geometric and cohomological methods in group theory, M.R. BRIDSON, P.H. KROPHOLLER & I.J. LEARY (eds)
- 359 Moduli spaces and vector bundles, L. BRAMBILA-PAZ, S.B. BRADLOW, O. GARCÍA-PRADA & S. RAMANAN (eds)
- 360 Zariski geometries, B. ZILBER
- 361 Words: Notes on verbal width in groups, D. SEGAL
- 362 Differential tensor algebras and their module categories, R. BAUTISTA, L. SALMERÓN & R. ZUAZUA
- 363 Foundations of computational mathematics, Hong Kong 2008, F. CUCKER, A. PINKUS & M.J. TODD (eds)
- 364 Partial differential equations and fluid mechanics, J.C. ROBINSON & J.L. RODRIGO (eds)
- 365 Surveys in combinatorics 2009, S. HUCZYNSKA, J.D. MITCHELL & C.M. RONEY-DOUGAL (eds)
- 366 Highly oscillatory problems, B. ENGQUIST, A. FOKAS, E. HAIRER & A. ISERLES (eds)
- 367 Random matrices: High dimensional phenomena, G. BLOWER
- 368 Geometry of Riemann surfaces, F.P. GARDINER, G. GONZÁLEZ-DIEZ & C. KOUROUNIOTIS (eds)
- 369 Epidemics and rumours in complex networks, M. DRAIEF & L. MASSOULIÉ
- 370 Theory of p-adic distributions, S. ALBEVERIO, A.YU. KHRENNIKOV & V.M. SHELKOVICH
- 371 Conformal fractals, F. PRZYTYCKI & M. URBAŃSKI
- 372 Moonshine: The first quarter century and beyond, J. LEPOWSKY, J. MCKAY & M.P. TUIE (eds)
- 373 Smoothness, regularity and complete intersection, J. MAJADAS & A. G. RODICIO
- 374 Geometric analysis of hyperbolic differential equations: An introduction, S. ALINHAC
- 375 Triangulated categories, T. HOLM, P. JØRGENSEN & R. ROUQUIER (eds)
- 376 Permutation patterns, S. LINTON, N. RUŠKUC & V. VATTER (eds)
- 377 An introduction to Galois cohomology and its applications, G. BERHUY
- 378 Probability and mathematical genetics, N. H. BINGHAM & C. M. GOLDIE (eds)
- 379 Finite and algorithmic model theory, J. ESPARZA, C. MICHAUX & C. STEINHORN (eds)
- 380 Real and complex singularities, M. MANOEL, M.C. ROMERO FUSTER & C.T.C WALL (eds)
- 381 Symmetries and integrability of difference equations, D. LEVI, P. OLVER, Z. THOMOVA & P. WINTERNITZ (eds)
- 382 Forcing with random variables and proof complexity, J. KRAJÍČEK

Cambridge University Press

978-1-107-46249-6 - O-Minimality and Diophantine Geometry: London Mathematical Society Lecture Note Series: 421

Edited by G. O. Jones and A. J. Wilkie

Frontmatter

[More information](#)

- 383 Motivic integration and its interactions with model theory and non-Archimedean geometry I, R. CLUCKERS, J. NICAISE & J. SEBAG (eds)
- 384 Motivic integration and its interactions with model theory and non-Archimedean geometry II, R. CLUCKERS, J. NICAISE & J. SEBAG (eds)
- 385 Entropy of hidden Markov processes and connections to dynamical systems, B. MARCUS, K. PETERSEN & T. WEISSMAN (eds)
- 386 Independence-friendly logic, A.L. MANN, G. SANDU & M. SEVENSTER
- 387 Groups St Andrews 2009 in Bath I, C.M. CAMPBELL *et al* (eds)
- 388 Groups St Andrews 2009 in Bath II, C.M. CAMPBELL *et al* (eds)
- 389 Random fields on the sphere, D. MARINUCCI & G. PECCATI
- 390 Localization in periodic potentials, D.E. PELINOVSKY
- 391 Fusion systems in algebra and topology, M. ASCHBACHER, R. KESSAR & B. OLIVER
- 392 Surveys in combinatorics 2011, R. CHAPMAN (ed)
- 393 Non-abelian fundamental groups and Iwasawa theory, J. COATES *et al* (eds)
- 394 Variational problems in differential geometry, R. BIELAWSKI, K. HOUSTON & M. SPEIGHT (eds)
- 395 How groups grow, A. MANN
- 396 Arithmetic differential operators over the p-adic integers, C.C. RALPH & S.R. SIMANCA
- 397 Hyperbolic geometry and applications in quantum chaos and cosmology, J. BOLTE & F. STEINER (eds)
- 398 Mathematical models in contact mechanics, M. SOFONEA & A. MATEI
- 399 Circuit double cover of graphs, C.-Q. ZHANG
- 400 Dense sphere packings: a blueprint for formal proofs, T. HALES
- 401 A double Hall algebra approach to affine quantum Schur–Weyl theory, B. DENG, J. DU & Q. FU
- 402 Mathematical aspects of fluid mechanics, J.C. ROBINSON, J.L. RODRIGO & W. SADOWSKI (eds)
- 403 Foundations of computational mathematics, Budapest 2011, F. CUCKER, T. KRICK, A. PINKUS & A. SZANTO (eds)
- 404 Operator methods for boundary value problems, S. HASSI, H.S.V. DE SNOO & F.H. SZAFRANIEC (eds)
- 405 Torsors, étale homotopy and applications to rational points, A.N. SKOROBOGATOV (ed)
- 406 Appalachian set theory, J. CUMMINGS & E. SCHIMMERLING (eds)
- 407 The maximal subgroups of the low-dimensional finite classical groups, J.N. BRAY, D.F. HOLT & C.M. RONEY-DOUGAL
- 408 Complexity science: the Warwick master's course, R. BALL, V. KOLOKOLTSOV & R.S. MACKAY (eds)
- 409 Surveys in combinatorics 2013, S.R. BLACKBURN, S. GERKE & M. WILDON (eds)
- 410 Representation theory and harmonic analysis of wreath products of finite groups, T. CECCHERINI-SILBERSTEIN, F. SCARABOTTI & F. TOLLI
- 411 Moduli spaces, L. BRAMBILA-PAZ, O. GARCÍA-PRADA, P. NEWSTEAD & R.P. THOMAS (eds)
- 412 Automorphisms and equivalence relations in topological dynamics, D.B. ELLIS & R. ELLIS
- 413 Optimal transportation, Y. OLLIVIER, H. PAJOT & C. VILLANI (eds)
- 414 Automorphic forms and Galois representations I, F. DIAMOND, P.L. KASSAEI & M. KIM (eds)
- 415 Automorphic forms and Galois representations II, F. DIAMOND, P.L. KASSAEI & M. KIM (eds)
- 416 Reversibility in dynamics and group theory, A.G. O'FARRELL & I. SHORT
- 417 Recent advances in algebraic geometry, C.D. HACON, M. MUSTAŢĂ & M. POPA (eds)
- 418 The Bloch-Kato conjecture for the Riemann zeta function, J. COATES, A. RAGHURAM, A. SAIKIA & R. SUJATHA (eds)
- 419 The Cauchy problem for non-Lipschitz semi-linear parabolic partial differential equations, J.C. MEYER & D.J. NEEDHAM
- 420 Arithmetic and geometry, L. DIEULEFAIT *et al* (eds)
- 421 O-minimality and Diophantine geometry, G.O. JONES & A.J. WILKIE (eds)

Cambridge University Press

978-1-107-46249-6 - O-Minimality and Diophantine Geometry: London Mathematical
Society Lecture Note Series: 421

Edited by G. O. Jones and A. J. Wilkie

Frontmatter

[More information](#)

Cambridge University Press

978-1-107-46249-6 - O-Minimality and Diophantine Geometry: London Mathematical
Society Lecture Note Series: 421

Edited by G. O. Jones and A. J. Wilkie

Frontmatter

[More information](#)

London Mathematical Society Lecture Note Series: 421

O-Minimality and Diophantine Geometry

Edited by

G. O. JONES

University of Manchester

A. J. WILKIE

University of Manchester



Cambridge University Press

978-1-107-46249-6 - O-Minimality and Diophantine Geometry: London Mathematical Society Lecture Note Series: 421

Edited by G. O. Jones and A. J. Wilkie

Frontmatter

[More information](#)

CAMBRIDGE
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning and research at the highest international levels of excellence.

www.cambridge.org

Information on this title: www.cambridge.org/9781107462496

© Cambridge University Press 2015

This publication is in copyright. Subject to statutory exception and to the provisions of relevant collective licensing agreements, no reproduction of any part may take place without the written permission of Cambridge University Press.

First published 2015

Printed in the United Kingdom by Clays, St Ives plc

A catalogue record for this publication is available from the British Library

Library of Congress Cataloguing in Publication data

Wilkie, A. J. (Alec J.)

O-minimality and diophantine geometry / G.O. Jones, University of Manchester, A.J. Wilkie, University of Manchester.

pages cm. – (London Mathematical Society lecture note series ; 421)

ISBN 978-1-107-46249-6 (pbk.)

1. Arithmetical algebraic geometry. 2. Model theory. 3. Geometry, Analytic.

I. Jones, G. O. (Gareth Owen). II. Title.

QA242.5.W55 2015

516.3'5–dc23 2014045023

ISBN 978-1-107-46249-6 Paperback

Cambridge University Press has no responsibility for the persistence or accuracy of URLs for external or third-party internet websites referred to in this publication, and does not guarantee that any content on such websites is, or will remain, accurate or appropriate.

Cambridge University Press

978-1-107-46249-6 - O-Minimality and Diophantine Geometry: London Mathematical Society Lecture Note Series: 421

Edited by G. O. Jones and A. J. Wilkie

Frontmatter

[More information](#)

Contents

	<i>Preface</i>	<i>page xi</i>
1	The Manin-Mumford Conjecture, an elliptic Curve, its Torsion Points & their Galois Orbits	1
	<i>P. Habegger</i>	
1	Overview	1
2	Elliptic Curves	4
3	Galois Orbits of Torsion Points and Heights	9
4	Application to the Manin-Mumford Conjecture	27
	Appendix An Inequality of Elkies for the Local Néron-Tate Height	30
2	Rational points on definable sets	41
	<i>A. J. Wilkie</i>	
1	Introduction	41
2	Some semi-algebraic geometry	42
3	O-minimal structures ([PS], [D1])	43
4	Some 1-dimensional o-minimal theory	46
5	Reparametrization (one variable case)	47
6	Proof of 1.1 in the 1-dimensional case	53
7	Some remarks on the proof of the general case of 1.1	58
8	Some higher-dimensional o-minimal theory	59
9	Reparametrization (many variable case)	61
3	Functional transcendence via o-minimality	66
	<i>Jonathan Pila</i>	
1	Algebraic independence	66

Cambridge University Press

978-1-107-46249-6 - O-Minimality and Diophantine Geometry: London Mathematical Society Lecture Note Series: 421

Edited by G. O. Jones and A. J. Wilkie

Frontmatter

[More information](#)

viii

Contents

2	Transcendental numbers	67
3	Schanuel's conjecture	69
4	Differential fields	70
5	Ax-Schanuel	71
6	"Ax-Lindemann"	75
7	The modular function	76
8	Modular Schanuel Conjecture	79
9	"Modular Ax-Schanuel"	80
10	"Modular Ax-Lindemann"	82
11	The general setting	82
12	Exponential Ax-Lindemann via o-minimality	84
13	Modular Ax-Lindemann via o-minimality	87
14	SC and CIT	91
15	Zilber-Pink	94
16	Zilber-Pink and Ax-Schanuel	95
4	Introduction to abelian varieties and the Ax–Lindemann–Weierstrass theorem	100
	<i>Martin Orr</i>	
1	Introduction	100
2	Abelian varieties	102
3	Complex tori	106
4	Riemann forms and polarisations	107
5	The moduli space of principally polarised abelian varieties	111
6	Complex multiplication	117
7	The Ax–Lindemann–Weierstrass theorem for abelian varieties	119
8	Relationship between semialgebraic and complex algebraic sets	121
9	Proof of the Ax–Lindemann–Weierstrass theorem for abelian varieties	123
5	The André–Oort conjecture via o-minimality	129
	<i>Christopher Daw</i>	
1	Introduction	129
2	Hermitian symmetric domains	131
3	Conjugacy classes	133
4	The Deligne torus	135
5	Hodge structures	135
6	Abelian varieties	137

Cambridge University Press

978-1-107-46249-6 - O-Minimality and Diophantine Geometry: London Mathematical Society Lecture Note Series: 421

Edited by G. O. Jones and A. J. Wilkie

Frontmatter

[More information](#)

<i>Contents</i>		ix
7	The Siegel upper half-space	140
8	Families of Hodge structures	140
9	The algebraic group	141
10	Shimura data	141
11	Congruence subgroups	143
12	Adeles	144
13	Neatness	144
14	Shimura varieties	145
15	Complex structure	145
16	Algebraic structure	146
17	Special subvarieties	147
18	Special points	147
19	Canonical model	148
20	The André-Oort conjecture	150
21	Reductions	150
22	Galois orbits	151
23	Realisations	152
24	Heights	152
25	Definability	154
26	Ax-Lindemann-Weierstrass	154
27	Pila-Wilkie	155
28	Final reduction	155
29	The Pila-Zannier strategy	155
6	Lectures on elimination theory for semialgebraic and subanalytic sets	159
	<i>A.J. Wilkie</i>	
1	Model Theoretic Generalities	160
2	The Real Field	164
3	Preliminary Remarks on Rings and Modules	166
4	Formal Power Series Rings	168
5	Adically Normed Rings	169
6	Formal Power Series in Many Variables	172
7	Convergent Power Series	176
8	More on Adically Normed Rings and Modules	181
9	The Denef–van den Dries Paper	185
7	Relative Manin-Mumford for abelian varieties	193
	<i>D. Masser</i>	

Cambridge University Press

978-1-107-46249-6 - O-Minimality and Diophantine Geometry: London Mathematical Society Lecture Note Series: 421

Edited by G. O. Jones and A. J. Wilkie

Frontmatter

[More information](#)

x

Contents

8	Improving the bound in the Pila-Wilkie theorem for curves	204
	<i>G. O. Jones</i>	
9	Ax-Schanuel and o-minimality	216
	<i>Jacob Tsimerman</i>	
1	Interpreting Ax-Schanuel Geometrically	216
2	An o-minimality proof of Ax-Schanuel	218

Cambridge University Press

978-1-107-46249-6 - O-Minimality and Diophantine Geometry: London Mathematical Society Lecture Note Series: 421

Edited by G. O. Jones and A. J. Wilkie

Frontmatter

[More information](#)

Preface

In July 2013 an LMS-EPSRC Short Instructional Course on ‘O-minimality and diophantine geometry’ was held in the School of Mathematics at the University of Manchester. This volume consists of lecture notes from the courses together with several other surveys. The motivation behind the short course was to introduce participants to some of the ideas behind Pila’s recent proof of the André-Oort conjecture for products of modular curves. The underlying ideas are similar to an earlier proof by Pila and Zannier of the Manin-Mumford conjecture (which has in fact long been a theorem, originally due to Raynaud) and combining the results of the various contributions here leads to a proof of this conjecture in certain cases. The basic strategy has three main ingredients: the Pila-Wilkie theorem, bounds on Galois orbits, and functional transcendence results. Each of the topics was the focus of a course. Wilkie discussed o-minimality and the Pila-Wilkie theorem without assuming any background in mathematical logic. (The argument given here is, in fact, slightly different from that given in the original paper, at least in the one-dimensional case.) Habegger’s course focused on the Galois bounds and on the completion of the proof (of certain cases of Manin-Mumford) from the various ingredients. And Pila’s lectures covered functional transcendence, also touching on various recent related work by Zilber. We have also included some further lecture notes by Wilkie containing a proof of the o-minimality of the expansion of the real field by restricted analytic functions, which is sufficient for the application of Pila-Wilkie to Manin-Mumford. At the short course there were also three guest lectures. Yafaev spoke on very recent breakthroughs on the functional transcendence side in the setting of general Shimura varieties. Masser spoke on some other results (‘relative Manin-Mumford’) that can be obtained by a similar strategy. Jones discussed improvements to the Pila-Wilkie theorem. Unfortunately, Yafaev was unable to contribute to this volume. During the week of the course, tutorials were given by Daw and Orr. For this volume,

Cambridge University Press

978-1-107-46249-6 - O-Minimality and Diophantine Geometry: London Mathematical Society Lecture Note Series: 421

Edited by G. O. Jones and A. J. Wilkie

Frontmatter

[More information](#)

Orr has written a survey of abelian varieties which contains a proof of the functional transcendence result necessary for the application in Habegger's course. Daw has contributed an introduction to Shimura varieties which we hope will prove valuable to those who wish to go on to study the general André-Oort conjecture. Finally, we are pleased to include a paper by Tsimerman in which he gives a proof of Ax's theorem on the functional case of Schanuel's conjecture via o-minimality.

We would like to thank the London Mathematical Society and the Engineering and Sciences Research Council for funding the course, and the School of Mathematics at the University of Manchester for hosting the meeting. And we are grateful to the speakers and tutors at the meeting and to the contributors to this volume.