‘An understanding of the remarkable properties of the Poisson process is essential for anyone interested in the mathematical theory of probability or in its many fields of application. This book is a lucid and thorough account, rigorous but not pedantic, and accessible to any reader familiar with modern mathematics at first-degree level. Its publication is most welcome.’

— J. F. C. Kingman, *University of Bristol*

‘I have always considered the Poisson process to be a cornerstone of applied probability. This excellent book demonstrates that it is a whole world in and of itself. The text is exciting and indispensable to anyone who works in this field.’

— Dietrich Stoyan, *TU Bergakademie Freiberg*

‘Last and Penrose’s *Lectures on the Poisson Process* constitutes a splendid addition to the monograph literature on point processes. While emphasising the Poisson and related processes, their mathematical approach also covers the basic theory of random measures and various applications, especially to stochastic geometry. They assume a sound grounding in measure-theoretic probability, which is well summarised in two appendices (on measure and probability theory). Abundant exercises conclude each of the twenty-two “lectures” which include examples illustrating their “course” material. It is a first-class complement to John Kingman’s essay on the Poisson process.’

— Daryl Daley, *University of Melbourne*

‘Pick n points uniformly and independently in a cube of volume n in Euclidean space. The limit of these random configurations as $n \to \infty$ is the Poisson process. This book, written by two of the foremost experts on point processes, gives a masterful overview of the Poisson process and some of its relatives. Classical tenets of the theory, like thinning properties and Campbell’s formula, are followed by modern developments, such as Liggett’s extra heads theorem, Fock space, permanental processes and the Boolean model. Numerous exercises throughout the book challenge readers and bring them to the edge of current theory.’

— Yuval Peres, *Principal Researcher, Microsoft Research, and Foreign Associate, National Academy of Sciences*

Lectures on the Poisson Process

The Poisson process, a core object in modern probability, enjoys a richer theory than is sometimes appreciated. This volume develops the theory in the setting of a general abstract measure space, establishing basic results and properties as well as certain advanced topics in the stochastic analysis of the Poisson process. Also discussed are applications and related topics in stochastic geometry, including stationary point processes, the Boolean model, the Gilbert graph, stable allocations and hyperplane processes. Comprehensive, rigorous, and self-contained, this text is ideal for graduate courses or for self-study, with a substantial number of exercises for each chapter. Mathematical prerequisites, mainly a sound knowledge of measure-theoretic probability, are kept in the background, but are reviewed comprehensively in an appendix. The authors are well-known researchers in probability theory, especially stochastic geometry. Their approach is informed both by their research and by their extensive experience in teaching at undergraduate and graduate levels.

GÜNTER LAST is Professor of Stochastics at the Karlsruhe Institute of Technology. He is a distinguished probabilist with particular expertise in stochastic geometry, point processes and random measures. He has coauthored a research monograph on marked point processes on the line as well as two textbooks on general mathematics. He has given many invited talks on his research worldwide.

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Lectures on the Poisson Process

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To Our Families

Contents

Preface

The Poisson process generates point patterns in a purely random manner. It plays a fundamental role in probability theory and its applications, and enjoys a rich and beautiful theory. While many of the applications involve point processes on the line, or more generally in Euclidean space, many others do not. Fortunately, one can develop much of the theory in the abstract setting of a general measurable space.

We have prepared the present volume so as to provide a modern textbook on the general Poisson process. Despite its importance, there are not many monographs or graduate texts with the Poisson process as their main point of focus, for example by comparison with the topic of Brownian motion. This is probably due to a viewpoint that the theory of Poisson processes on its own is too insubstantial to merit such a treatment. Such a viewpoint now seems out of date, especially in view of recent developments in the stochastic analysis of the Poisson process. We also extend our remit to topics in stochastic geometry, which is concerned with mathematical models for random geometric structures [4, 5, 23, 45, 123, 126, 147]. The Poisson process is fundamental to stochastic geometry, and the applications areas discussed in this book lie largely in this direction, reflecting the taste and expertise of the authors. In particular, we discuss Voronoi tessellations, stable allocations, hyperplane processes, the Boolean model and the Gilbert graph.

Besides stochastic geometry, there are many other fields of application of the Poisson process. These include Lévy processes [10, 83], Brownian excursion theory [140], queueing networks [6, 149], and Poisson limits in extreme value theory [139]. Although we do not cover these topics here, we hope nevertheless that this book will be a useful resource for people working in these and related areas.

This book is intended to be a basis for graduate courses or seminars on the Poisson process. It might also serve as an introduction to point process theory. Each chapter is supposed to cover material that can be presented (at least in principle) in a single lecture. In practice, it may not always be possible to get through an entire chapter in one lecture; however, in most chapters the most essential material is presented in the early part of the chapter, and the later part could feasibly be left as background reading if necessary. While it is recommended to read the earlier chapters in a linear order at least up to Chapter 5, there is some scope for the reader to pick and choose from the later chapters. For example, a reader more interested in stochastic geometry could look at Chapters 8–11 and 16–17. A reader wishing to focus on the general abstract theory of Poisson processes could look at Chapters 6, 7, 12, 13 and 18–21. A reader wishing initially to take on slightly easier material could look at Chapters 7–9, 13 and 15–17.

The book divides loosely into three parts. In the first part we develop basic results on the Poisson process in the general setting. In the second part we introduce models and results of stochastic geometry, most but not all of which are based on the Poisson process, and which are most naturally developed in the Euclidean setting. Chapters 8, 9, 10, 16, 17 and 22 are devoted exclusively to stochastic geometry while other chapters use stochastic geometry models for illustrating the theory. In the third part we return to the general setting and describe more advanced results on the stochastic analysis of the Poisson process.

Our treatment requires a sound knowledge of measure-theoretic probability theory. However, specific knowledge of stochastic processes is not assumed. Since the focus is always on the probabilistic structure, technical issues of measure theory are kept in the background, whenever possible. Some basic facts from measure and probability theory are collected in the appendices.

When treating a classical and central subject of probability theory, a certain overlap with other books is inevitable. Much of the material of the earlier chapters, for instance, can also be found (in a slightly more restricted form) in the highly recommended book [75] by J.F.C. Kingman. Further results on Poisson processes, as well as on general random measures and point processes, are presented in the monographs [6, 23, 27, 53, 62, 63, 69, 88, 107, 134, 139]. The recent monograph Kallenberg [65] provides an excellent systematic account of the modern theory of random measures. Comments on the early history of the Poisson process, on the history of the main results presented in this book and on the literature are given in Appendix C.

In preparing this manuscript we have benefited from comments on earlier versions from Daryl Daley, Fabian Gieringer, Christian Hirsch, Daniel Hug, Olav Kallenberg, Paul Keeler, Martin Möhle, Franz Nestmann, Jim

Pitman, Matthias Schulte, Tomasz Rolski, Dietrich Stoyan, Christoph Thäle, Hermann Thorisson and Hans Zessin, for which we are most grateful. Thanks are due to Franz Nestmann for producing the figures. We also wish to thank Olav Kallenberg for making available to us an early version of his monograph [65].

Günter Last
Mathew Penrose

Symbols

$\mathbb{Z} = \{0, 1, -1, 2, -2, \dots\}$	set of integers
$\mathbb{N} = \{1, 2, 3, 4, \dots\}$	set of positive integers
$\mathbb{N}_0 = \{0, 1, 2, \dots\}$	set of non-negative integers
$\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$	extended set of positive integers
$\overline{\mathbb{N}}_0 = \mathbb{N}_0 \cup \{\infty\}$	extended set of non-negative integers
$\mathbb{R} = (-\infty, \infty), \mathbb{R}_+ = [0, \infty)$	real line (resp. non-negative real half-line)
$\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$	extended real line
$\overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{\infty\} = [0, \infty]$	extended half-line
$\mathbb{R}(\mathbb{X}), \mathbb{R}_+(\mathbb{X})$	$\mathbb{R}$-valued (resp. $\mathbb{R}_+$-valued) measurable functions on $\mathbb{X}$
$\overline{\mathbb{R}}(\mathbb{X}), \overline{\mathbb{R}}_+(\mathbb{X})$	$\overline{\mathbb{R}}$-valued (resp. $\overline{\mathbb{R}}_+$-valued) measurable functions on $\mathbb{X}$
u^+, u^-	positive and negative part of an $\overline{\mathbb{R}}$-valued function u
$a \wedge b, a \vee b$	minimum (resp. maximum) of $a, b \in \overline{\mathbb{R}}$
$\mathbf{1}\{\cdot\}$	indicator function
$a^{\oplus} := \mathbf{1}\{a \neq 0\}a^{-1}$	generalised inverse of $a \in \mathbb{R}$
$\operatorname{card} A = \|A\|$	number of elements of a set A
$[n]$	$\{1, \dots, n\}$
Σ_n	group of permutations of $[n]$
Π_n, Π_n^*	set of all partitions (resp. subpartitions) of $[n]$
$(n)_k = n \cdots (n - k + 1)$	descending factorial
δ_x	Dirac measure at the point x
$\mathbf{N}_{<\infty}(\mathbb{X}) \equiv \mathbf{N}_{<\infty}$	set of all finite counting measures on $\mathbb{X}$
$\mathbf{N}(\mathbb{X}) \equiv \mathbf{N}$	set of all countable sums of measures from $\mathbf{N}_{<\infty}$
$\mathbf{N}_l(\mathbb{X}), \mathbf{N}_s(\mathbb{X})$	set of all locally finite (resp. simple) measures in $\mathbf{N}(\mathbb{X})$
$\mathbf{N}_{ls}(\mathbb{X}) := \mathbf{N}_l(\mathbb{X}) \cap \mathbf{N}_s(\mathbb{X})$	set of all locally finite and simple measures in $\mathbf{N}(\mathbb{X})$
$x \in \mu$	short for $\mu\{x\} = \mu(\{x\}) > 0, \mu \in \mathbf{N}$
ν_B	restriction of a measure ν to a measurable set B

$\mathcal{B}(\mathbb{X})$	Borel σ-field on a metric space $\mathbb{X}$
$\mathcal{X}_b$	bounded Borel subsets of a metric space $\mathbb{X}$
$\mathbb{R}^d$	Euclidean space of dimension $d \in \mathbb{N}$
$\mathcal{B}^d := \mathcal{B}(\mathbb{R}^d)$	Borel σ-field on $\mathbb{R}^d$
λ_d	Lebesgue measure on $(\mathbb{R}^d, \mathcal{B}^d)$
$\|\cdot\|$	Euclidean norm on $\mathbb{R}^d$
$\langle \cdot, \cdot \rangle$	Euclidean scalar product on $\mathbb{R}^d$
$\mathcal{C}^d, \mathcal{C}^{(d)}$	compact (resp. non-empty compact) subsets of $\mathbb{R}^d$
$\mathcal{K}^d, \mathcal{K}^{(d)}$	compact (resp. non-empty compact) convex subsets of $\mathbb{R}^d$
$\mathcal{R}^d$	convex ring in $\mathbb{R}^d$ (finite unions of convex sets)
$K + x, K - x$	translation of $K \subset \mathbb{R}^d$ by x (resp. $-x$)
$K \oplus L$	Minkowski sum of $K, L \subset \mathbb{R}^d$
$V_0, \dots, V_d$	intrinsic volumes
$\phi_i = \int V_i(K)\, \mathbb{Q}(dK)$	i-th mean intrinsic volume of a typical grain
$B(x, r)$	closed ball with centre x and radius $r \geq 0$
$\kappa_d = \lambda_d(B^d)$	volume of the unit ball in $\mathbb{R}^d$
$<$	strict lexicographical order on $\mathbb{R}^d$
$l(B)$	lexicographic minimum of a non-empty finite set $B \subset \mathbb{R}^d$
$(\Omega, \mathcal{F}, \mathbb{P})$	probability space
$\mathbb{E}[X]$	expectation of a random variable X
$\mathbb{V}\mathrm{ar}[X]$	variance of a random variable X
$\mathbb{C}\mathrm{ov}[X, Y]$	covariance between random variables X and Y
L_η	Laplace functional of a random measure η
$\overset{d}{=}, \overset{d}{\to}$	equality (resp. convergence) in distribution