INTRODUCTION.

BEFORE entering upon the subject proper of the present work, a short preliminary discussion of those parts of synthetic geometry which have the closest connexion with line geometry, has been inserted here for convenience of reference.

i. **Double Ratio.** For four points $ABCD$ lying on a straight line the number $\frac{AB}{BC} \div \frac{AD}{DC}$ is called the Double Sectional Ratio or Double Ratio of the points, the *sense* of the segments AB &c. being taken into consideration; the terms Anharmonic Ratio and Cross Ratio are also used to designate this quantity, which is usually denoted by $(ABCD)$.

The orders in which the points may be taken are 24 in number, but there are only six different Double Ratios of the four points, for we find that any two orders which differ by a double interchange of two points have their double ratios equal, *e.g.*

$$(BADC) = \frac{BA}{AD} \div \frac{BC}{CD} = \frac{AB}{BC} \div \frac{AD}{DC} = (ABCD);$$

so that $\quad (ABCD) = (BADC) = (CDAB) = (DCBA).$

Secondly, if two non-consecutive members of a double ratio be interchanged the double ratio is *inverted*, *e.g.*

$$(ADCB) = \frac{AD}{DC} \div \frac{AB}{BC} = \frac{1}{(ABCD)}.$$

Thirdly, the sum of the double ratios for two orders which differ in their second and third members is *unity*, *e.g.*

$$(ABCD) = \frac{AB \,.\, DC}{BC \,.\, AD}, \quad (ACBD) = \frac{AC \,.\, DB}{CB \,.\, AD} = \frac{CA \,.\, DB}{BC \,.\, AD},$$

and since

$$BC + CA + AB = 0, \quad AD = BD + AB, \quad AD = CD - CA,$$

therefore

$$BC \,.\, AD + CA\,(BD + AB) + AB\,(CD - CA) = 0,$$

or

$$BC \,.\, AD + CA \,.\, BD + AB \,.\, CD = 0,$$

hence

$$(ABCD) + (ACBD) = 1.$$

So that denoting $(ABCD)$ by λ, we have

$$(ADCB) = \frac{1}{\lambda}, \qquad (ACBD) = 1 - \lambda, \qquad (ADBC) = \frac{1}{1-\lambda},$$

$$(ABDC) = 1 - \frac{1}{1-\lambda} = \frac{\lambda}{\lambda - 1}, \qquad (ACDB) = \frac{\lambda - 1}{\lambda}.$$

All the other double ratios have one of these six values. If the value of the double ratio is -1 the points are said to be Harmonic; in this case since

$$\frac{AB}{BC} + \frac{AD}{DC} = 0,$$

hence

$$AB\,(AC - AD) + AD\,(AC - AB) = 0,$$

or

$$\frac{2}{AC} = \frac{1}{AB} + \frac{1}{AD};$$

whence it is easily found that if O is the mid-point of AC,

$$OC^2 = OB \,.\, OD.$$

[It should be noticed that in this case the points $ABCD$ are arranged consecutively.]

ii. **Correspondence.** If between the points of a straight line a connexion is established such that to each point of the line corresponds one and only one point of the line, there is said to exist a "one-one" correspondence, or correlation, between its points. If x is the distance of any point P of the line from a fixed point O of the line, this correlation is defined by an equation of the form

$$Axx' + Bx + Cx' + D = 0,$$

where x' is the distance from O of the point P' which corresponds to P. It follows from this equation that

$$x = -\frac{Cx' + D}{Ax' + B}$$

and hence if P, Q, R, S are four points and P', Q', R', S' their corresponding points,

$$(PQRS) = (P'Q'R'S');$$

for $\quad PQ = x_2 - x_1 = (x_2' - x_1')\,(AD - BC)/(Ax_1' + B)\,(Ax_2' + B),$

$\quad SR = x_3 - x_4 = (x_3' - x_4')\,(AD - BC)/(Ax_3' + B)\,(Ax_4' + B)$, &c.,

hence

$$(PQRS) = \frac{(x_1 - x_2)(x_4 - x_3)}{(x_2 - x_3)(x_1 - x_4)} = \frac{(x_1' - x_2')(x_4' - x_3')}{(x_2' - x_3')(x_1' - x_4')} = (P'Q'R'S').$$

It follows that if three pairs of points of the line be associated the correspondence is determined, for if P and P', Q and Q', R and R' be made to correspond, then the point S' which corresponds to any fourth point S is determined by the equality

$$(PQRS) = (P'Q'R'S').$$

iii. **United Points.** The coincidence of a point and its corresponding point will occur twice, for putting $x' = x$ we have

$$Ax^2 + (B + C)x + D = 0,$$

thus in every (1, 1) correspondence there are two "united" points (real or imaginary).

If the point midway between the united points (E, E') be the point O from which the distances are measured, we must have $B + C = 0$, and the equation defining the correspondence is of the form $Axx' + B(x - x') + D = 0$, while the distance α of either united point from O is given by the equation $A\alpha^2 + D = 0$; combining these two equations and writing κ for $\frac{B}{A}$, the equation of correspondence becomes

$$xx' + \kappa(x - x') - \alpha^2 = 0,$$

which may be written in the form

$$(x + \alpha)(x' - \alpha) = (\alpha + \kappa)(x' - x),$$

hence
$$\frac{2\alpha}{\alpha + \kappa} = \frac{(x - x')(-2\alpha)}{(x' - \alpha)(x + \alpha)} = \frac{PP' \,.\, E'E}{P'E \,.\, PE'} = (PP'EE'),$$

thus the double ratio of a point, its corresponding point, and the united points is *constant*. The correspondence is therefore determined if its united points and one pair of corresponding points are given.

iv. **Involution.** If $B = C$ the relation between x and x' is symmetrical, and hence if P' corresponds to any point P then will P correspond to P', and the points of the line form "closed systems" of two points. The correspondence is in this case called an Involution. The equation which connects corresponding points being now

$$Axx' + B(x + x') + D = 0,$$

it may be written

$$A\left(x + \frac{B}{A}\right)\left(x' + \frac{B}{A}\right) = \frac{B^2 - AD}{A},$$

or if y and y' are the respective distances of P and P' from the point whose distance from O is $-\dfrac{B}{A}$,

$$yy' = \frac{B^2 - AD}{A^2}.$$

If $B^2 > AD$ there are two *real* points each of which coincides with its corresponding point, viz. those given by the equation

$$y = \pm \frac{\sqrt{B^2 - AD}}{A},$$

so that if these points be E and E' and M their middle point (the origin for the y's) and P, P' any pair of corresponding points

$$MP \,.\, MP' = ME^2.$$

This shows that the two "double" points E and E' of the involution form with any pair of corresponding points *a harmonic range.*

v. **Harmonic Involutions.** If in the two involutions on the same line determined respectively by

$$xx' + A(x + x') + B = 0,$$
$$yy' + C(y + y') + D = 0,$$

the double points of one form a pair in the other, *i.e.* are harmonic conjugates to the double points of the other, it is clear that

$$B + D - 2AC = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (a).$$

The Involutions are then said to be "harmonic" to each other. In this case, if to two points P and P' which are conjugate in the first Involution the conjugate points in the second Involution are Q and Q' respectively, Q and Q' are themselves conjugate in the first Involution; for by hypothesis

$$OQ = -\frac{C \,.\, OP + D}{OP + C}, \quad OQ' = -\frac{C \,.\, OP' + D}{OP' + C},$$

hence $\quad (OP + C)(OP' + C)(OQ \,.\, OQ' + A \,.\, \overline{OQ + OQ'} + B)$

$= (C \,.\, OP + D)(C \,.\, OP' + D) - A\,(\overline{C \,.\, OP + D} \,.\, \overline{OP' + C}$

$\qquad + \overline{C \,.\, OP' + D} \,.\, \overline{OP + C}) + B\,(OP + C)(OP' + C)$

$= (C^2 - D)(OP \,.\, OP' + A \,.\, \overline{OP + OP'} + B)$; from (a),

$= 0$, which proves the result stated.

vi. **Correspondences on different lines.** We shall now consider correspondences between the points of two different lines and the ruled surfaces (or plane curves) obtained as loci or envelopes of lines joining corresponding points. In what follows use will be made of the obvious fact that when a correspondence is established between the points of a line and these points are joined to any external point by a plane pencil of lines, a similar correspondence is thereby established between the lines of the pencil; similarly a correspondence established on a line gives rise to a correspondence between the planes of *any* pencil of planes (*i.e.* planes having a common line of intersection).

When a (1, 1) *correspondence exists between the points of two lines* in the same plane, the joins of pairs of corresponding points envelope a curve of the second class;* for joining any point P of the plane to the points of the two lines a (1, 1) correspondence is established between the lines of the pencil centre P; since there are two *united lines* in this correspondence, through P will pass two and only two lines which connect a pair of corresponding points. To this envelope the two given lines are themselves tangents. A special case arises when the point of intersection O of the two given lines *corresponds to itself*, *i.e.* regarded as a point of the first line has itself as corresponding point in the second line; in this case, of the two lines through P which join corresponding points *one* coincides with PO and therefore passes through the fixed point O; the envelope of lines joining corresponding points breaks up into two points, O and one other point C, the two rows of points are said to be in *perspective*, and the point C through which pass all lines joining corresponding points is called the "centre of perspective."

The corresponding theorem afforded by the Principle of Duality is, *if between the lines of two plane pencils a* (1, 1) *correspondence exists, the locus of intersection of corresponding lines is a curve of the second order passing through the centres of the pencils;* for on any line the two pencils determine a (1, 1) correspondence of points, the two double points of which are the points of intersection of the line with the required locus. A special case arises when the line joining the centres of the pencils corresponds to itself; in this case, of the two double points on any line, *one* lies on the line joining the centres of the pencils, *i.e.* the locus of intersection of corresponding lines of the two pencils breaks up into the line joining the centres of the pencils and one other line c, the two pencils are said to be in *perspective*, and the line c which contains

* The x of Art. ii here refers to a point P of one line, and x' to its corresponding point P' on the other line.

the intersections of corresponding lines is called the *axis of perspective.*

vii. *A* (1, 1) *correspondence between the lines of two pencils* (*or the points of two lines*) *in one plane is established when to three elements of one are assigned as correspondents three elements of the other;* for if three lines of one pencil SA, SB, SC meet any given line p in A, B, and C, and three corresponding lines $S'A'$, $S'B'$, $S'C'$ of the other pencil meet the same line in A', B', C', then (Art. ii) the three pairs of corresponding points AA', BB', CC' determine a correlation on p, hence if any other line of the first pencil meets p in P the corresponding line $S'P'$ of the other pencil is determined.

viii. *The joins of corresponding points on two non-intersecting lines form one set of generators of a quadric, that is, a Regulus*;* for the points of the two lines u and v establish on the pencil of planes whose axis is any line l a (1, 1) correspondence, viz., if P and P' are corresponding points on u and v, to the plane (P, l) corresponds the plane (P', l), each of the two double planes of this correspondence will meet the lines u, v in a pair of corresponding points, hence *two and only two* lines joining corresponding points on u and v will meet l, and any line l will meet the locus of lines which join corresponding points on u and v in *two* points. The two given lines belong to the other system of generators of the quadric determined by the Regulus. It is to be noticed that the lines of a Regulus determine on *any* two lines of the other system two rows of points having a (1, 1) correspondence; and four given generators determine on a variable generator of the opposite system four points having a constant Double Ratio.

The Principle of Duality gives the theorem, *the locus of intersection of corresponding planes of two pencils of planes connected by a* (1, 1) *correspondence is a Regulus;* for the two pencils of planes determine on any line l a (1, 1) correspondence of points, hence corresponding planes will only meet on l at the double points of this correspondence.

The following properties of a quadric should be observed:

First, if A, B, C, D are any four points on a generator and a, b, c, d the tangent planes thereat, $(ABCD) = (abcd)$.

* In the sequel the word Regulus is restricted to mean 'one set of generators of a quadric surface,' the other set of generators is called, in reference to it, the 'complementary' regulus. The word 'demi-quadrique' is used by Koenigs in this sense.

This follows from taking any other generator of the same system which meets a, b, c, d in A', B', C', D' respectively, then AA', BB', CC', DD' are generators and therefore from what precedes $(ABCD)=(A'B'C'D')$, while $(A'B'C'D')=(abcd)$, hence $(abcd)=(ABCD)$;

Second, if $ABCD$ are any given points on the quadric and x any generator, the double ratio of the four planes xA, xB, xC, xD is *constant*; for if the generators through A, B, C, D of the opposite system to x meet x in $PQRS$ respectively, xA, xB, xC, xD are the tangent planes at P, Q, R, S respectively, therefore $(PQRS)=(xA, xB, xC, xD)$, but $(PQRS)$ being the double ratio of points of section by a generator of the four given generators through A, B, C, D is *constant*.

ix. **Correspondence between the points of a conic and the lines of a plane pencil.** *If a* (1, 1) *correspondence exists between the points of a conic and the lines of a plane pencil centre S, there are three points on the conic, of which one at least is real, through which pass their corresponding lines;* for take any point S' on the conic f^2 and join it to the points of f^2, then between the lines of the pencils centres S and S', a (1, 1) correspondence is established and the intersection of corresponding lines being a conic ϕ^2 which meets f^2 in four points, of which S' is one, it is seen that there are three points on f^2 through which pass the respective corresponding lines of the pencil centre S.

If between a pencil of planes and the lines of a regulus a (1, 1) correspondence exists, and if the section of the pencil and the regulus by any plane be taken, the last result shows that in this plane there are three points in which a plane of the pencil meets its corresponding line of the regulus, hence the locus of the intersection of corresponding generators and planes is a curve of which three points lie in any plane or a *twisted cubic*.

x. **Involution on a conic.** If a (1, 1) correspondence exists between the points of a conic, to a point P will correspond a point Q and to Q a point R in general different from P, thus through Q pass two lines which join corresponding points, and this being the case for each point of the conic, the envelope of lines joining corresponding points is a curve of the *second class*. If however the correspondence is involutory, *i.e.* if to Q corresponds P, the envelope becomes of the first class or the lines joining corresponding points are concurrent; if U is their common point, U is called the *centre* of the involution.

Condition for Involution. A (1, 1) correspondence on the same conic is an Involution when to a point A *there corresponds doubly* a point A_1; for let B and B_1 be two other corresponding points, so that to AA_1B correspond A_1AB_1, let U be the intersection of AA_1 and BB_1 and u its polar line with respect to the conic.

The pencils $B_1(A_1AB...)$, $B(AA_1B_1...)$ are in (1, 1) correspondence and since they have the line BB_1 as self-corresponding line the pencils are in perspective and the locus of intersection of corresponding lines of the pencils is a straight line which must be u.

So that any line BC meets its corresponding line B_1C_1 in u, hence C, U, and C_1 must be collinear and therefore since to C corresponds C_1, to C_1 will correspond C, or the correspondence is an *Involution.*

xi. **Corresponding Sheaves.** The assemblage of ∞^2 lines which pass through one point are said to form a *sheaf*; the same name is given to all the planes through a point. If the sheaf of planes through any point S is connected by a (1, 1) correspondence with the sheaf of planes through a point S', so that to the intersection of two planes through S corresponds the intersection of the corresponding planes through S', the sheaves are said to be *collinear.*

xii. **Systems of Lines.** *The system of lines formed by the intersection of pairs of corresponding planes of two collinear sheaves is of the* first order, *i.e. through any point P there passes* one *line of the system;* for join S to P, then through S' there is one corresponding line $S'P'$ and the two pencils of planes for SP and $S'P'$, being connected by a (1, 1) correspondence, have as locus of intersection of corresponding pairs a *regulus* (Art. viii), of which *one line* passes through P; SP and $S'P'$ intersect all the lines of this regulus.

But if the line $S'P$ corresponds to SP, i.e. if the two corresponding lines intersect in P, the above regulus becomes a quadric cone of vertex P and *all* the generators of the cone belong to the system of lines; P is then called a "singular" point of the system of lines.

[It should be noticed that SP and $S'P$ are both generators of this cone, for to the plane SPS' of the pencil whose axis is SP corresponds a plane through $S'P$, hence $S'P$ is a line of intersection of two corresponding planes; similarly for SP.]

If the plane SPS' corresponds to itself the cone becomes a plane; for take any plane α through SS', the two pencils of planes through SP and $S'P$ meet α in corresponding lines SQ, $S'Q$ and the locus of Q is a *line* p, since for the pencils (S, α), (S', α) the line SS' corresponds to itself, (because to the plane SPS' the plane $S'PS$ corresponds), hence the lines of the system through P, being the lines PQ, lie in the plane (P, p).

Every regulus or cone of the system passes through each singular point, since to a plane through such a point P corresponds a plane which must also pass through P.

It has been shown that through each point which is not a singular point one line of the system passes; it remains to determine the *class* of the system of lines, *i.e.* the number in any plane.

I. *When the correspondence is such that SS' is a self-corresponding line the class is unity;* for if P is a singular point, to the plane PSS' corresponds the plane $PS'S$, *i.e.* this plane corresponds to itself and since SS' corresponds to itself the locus of intersection of corresponding lines in it is a line p of which every point is a singular point, therefore corresponding planes through S and S' meet p in the same point. Singular points not on p are seen for a similar reason to lie on another line p' which is intersected by all lines of the system. Thus the system of lines is formed by all the lines which meet the two lines p and p'; in any given plane there is one line of the system, viz. the join of the points in which the plane meets p and p'. There are two self-corresponding planes in the sheaves, viz. (SS', p), (SS', p').

II. *When the correspondence is such that the sheaves have only one self-corresponding plane, the class of the system is two;* for in this plane there is a set of singular points lying on a conic c^2 through S and S', and at a point P of c^2 the cone of P is a plane (see above); the planes of two such points P and Q meet in a line v every point of which is singular, for through any point R of v there are *two* lines, *i.e.* RP, RQ, belonging to the system and therefore an infinite number, also since v meets the given plane it intersects c^2. The lines of the system are hence the lines joining the points of v to those of c^2, the lines in any plane α are the joins of the point (v, α) to the points (c^2, α), *i.e.* two in number.

III. *When the collinear sheaves have no self-corresponding element the class of the system is three;* for if p is the intersection of two corresponding planes, to the pencil of lines (S, p) corresponds the pencil of lines (S', p) and these two pencils determine on p a

(1, 1) correspondence; if P and Q are its united points they are singular points of the system of lines and on P there are *no other* singular points; also if P is any singular point the plane PSS' contains the three singular points P, S, S' and no other, since if in it another singular point existed the plane PSS' would be self-corresponding as containing two pairs of corresponding lines; again since the cones of any two singular points P and P' each contain all the singular points, the locus of singular points consists of the partial intersection of two quadric cones having the generator PP' in common, *i.e.* a *twisted cubic**. The lines of the system consist of the chords of this cubic and those in a plane are the joins of the intersections of the plane and the cubic, three in number.

The following property of the twisted cubic is of importance:

If x is any chord of the curve and A, B, C, D given points on the curve the four planes xA, xB, xC, xD have a constant double ratio; for if any point P of the curve be joined to the other points of the curve we have a "cone of the system" and if x be any generator of this cone (xA, xB, xC, xD) is constant, since this merely expresses the property that the double ratio of the lines joining a variable point on a conic to four fixed points on the conic is constant, hence for *all* the chords of the cubic through P the theorem is true and if Q be *any other* point of the cubic since the cones for P and Q have one generator common, the theorem is also true for Q.

The locus of intersection of three corresponding planes of three pencils of planes which have mutually a (1, 1) *correspondence, is a twisted cubic of which the axes of the pencils are chords;* for the lines of intersection of corresponding planes of two of the pencils form a regulus and to each pair of corresponding planes there corresponds a plane of the third pencil, hence to each line of the regulus one plane of the third pencil corresponds, and it was seen (Art. ix) that the locus of intersection of corresponding members of a regulus and pencil of planes is a twisted cubic.

xiii. **Collinear Plane Systems.** By aid of two collinear sheaves a (1, 1) correspondence can be established between the points of any two planes, *i.e.* if any line of the sheaf centre S meets one plane in P and the corresponding line of the sheaf centre S' meets the other plane in P', then P and P' are a pair of corresponding points in the two planes. Further if the two planes are superposed on each other a (1, 1) correspondence of points of this plane arises, so that to each point of the plane correspond two points P', P'' according as P is supposed to belong to *one* of these (indefinitely near) planes or to the *other*; the points of the plane

* See Salmon's *Geometry of Three Dimensions*, third edition, p. 304.