1
Origins

To motivate our study of reversibility, we describe how the concept originates in dynamical systems, finite group theory, and in a subject known as *hidden dynamics*. Full details of these topics are beyond the scope of this book, and none of the material in this chapter is needed later on.

1.1 Origins in dynamical systems

Here we discuss several examples of reversibility in the study of conservative dynamical systems.

1.1.1 The harmonic oscillator

The simple pendulum is approximately modelled by the harmonic oscillator: the system in which a particle on the real line $\mathbb{R}$ is attracted to the origin by a force directly proportional to its distance from the origin. This system also models a weight suspended from a spring, oscillating about its equilibrium position (in which case the relationship between the force and distance is given by Hooke's law). Newton's second law states that the rate of change of the momentum of a body is equal to the force applied to it. Momentum is mass times velocity, so this gives the differential equations

$$\begin{aligned} \frac{dp}{dt} &= -\kappa q, \\ \frac{dq}{dt} &= \frac{p}{m}, \end{aligned} \tag{1.1}$$

where q represents the position of the particle, p its momentum (both p and q are functions of time t), κ is the constant of proportionality between the force and the distance to the origin, and m is the particle's mass.

It follows at once that the quantity

$$H(q,p) = \frac{p^2}{2m} + \frac{\kappa q^2}{2},$$

called its *Hamiltonian* (which is, physically, the energy of the system, given by the sum of its kinetic and potential energy), has derivative zero with respect to time, and hence is constant along trajectories. It follows that the trajectories are the concentric ellipses $H(q,p) = E$, for constant $E \geqslant 0$.

Consider the map $\tau : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\tau(q,p) = (q,-p)$. Evidently, $\tau \circ \tau = \mathbb{1}$, the identity map. A simple calculation establishes the following result.

Lemma 1.1 *If $(q(t), p(t))$ is a solution of the differential equations* (1.1), *then so is $\tau(q(-t), p(-t))$.* □

This lemma is usually expressed as saying that τ is a *time-reversal symmetry* of the system.

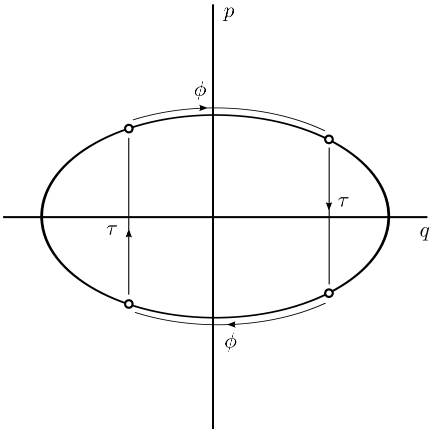


Figure 1.1 Time-reversal symmetry of the harmonic oscillator

Let $t \mapsto (q(t), p(t))$ represent the solution of (1.1) subject to the initial conditions $(q(0), p(0)) = (q_0, p_0)$, where (q_0, p_0) is some pair in $\mathbb{R}^2$. We define $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ to be the *time-one step of the system*, given by $\phi(q_0, p_0) = (q(1), p(1))$. Then

$$\phi \circ \tau \circ \phi \circ \tau = \mathbb{1},$$

or

$$\tau \circ \phi \circ \tau = \phi^{-1}, \tag{1.2}$$

the inverse map of ϕ (see Figure 1.1).

1.1.2 The n-body problem

The above behaviour is not particular to the harmonic oscillator. We can make similar observations whenever the Hamiltonian $H(q,p)$ of a dynamical system is quadratic in the momentum variable p.

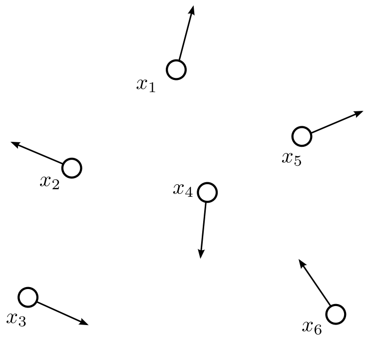


Figure 1.2 The n-body problem

Consider, for instance, the problem of n point bodies moving under their mutual gravitational attraction, illustrated in Figure 1.2. If we denote the masses by m_i and the positions by $x_i : \mathbb{R} \to \mathbb{R}^3$ $(i = 1,\dots,n)$, then in Newtonian form the equations of motion are

$$\frac{d}{dt}\left(m_i \frac{dx_i}{dt}\right) = \sum_{\substack{r=1 \\ r\neq i}}^{n} \frac{Gm_i m_r}{|x_r - x_i|^2}\left(\frac{x_r - x_i}{|x_r - x_i|}\right),$$

where G is the gravitational constant. Let $x_i = (x_{i1}, x_{i2}, x_{i3})$ for $i = 1,\dots,n$ and, for $j = 1,2,3$, let

$$\mu_{3i-3+j} = m_i, \quad q_{3i-3+j} = x_{ij}, \quad p_{3i-3+j} = m_i \frac{dx_{ij}}{dt}.$$

We also define

$$K(p) = \sum_{r=1}^{3n} \frac{p_r^2}{2\mu_r}, \quad V(q) = -\frac{1}{2} \sum_{\substack{r,s=1 \\ r\neq s}}^{n} \frac{Gm_r m_s}{|x_r - x_s|},$$

$$H(q,p) = K(p) + V(q),$$

where $p = (p_1, \dots, p_{3n})$ and $q = (q_1, \dots, q_{3n})$. Then the equations of motion become

$$\frac{dq_k}{dt} = \frac{\partial H}{\partial p_k},$$
$$\frac{dp_k}{dt} = -\frac{\partial H}{\partial q_k},$$

for $k = 1, \dots, 3n$. We have, as before, that $H(q,-p) = H(q,p)$, and that if $(q(t), p(t))$ is a solution, then so is $(q(-t), -p(-t))$.

This system has singularities when $n > 1$, some corresponding to collisions, and, for $n \geqslant 4$, some corresponding to other singularities [248]. Let us consider not the full phase space $\mathbb{R}^{3n} \times \mathbb{R}^{3n}$, but the subset X obtained by removing all orbits that end in a singularity, and all orbits that when run backwards end in a singularity. (By running an orbit $(q(t), p(t))$ backwards, we mean taking the orbit $(q(-t), -p(-t))$.) We remark that X is nonempty, but its structure is not fully understood to date [84].

Again, we can define $\tau(q,p) = (q,-p)$ and $\phi : X \to X$ to be the time-one step of the system, so that (1.2) holds.

1.1.3 Billiards

Consider billiards on an arbitrary smoothly-bounded, strictly-convex table without pockets. Let Γ denote the boundary. We ignore the motion in which the ball

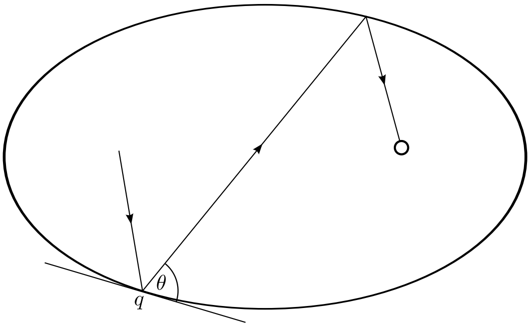


Figure 1.3 Trajectory of a billiard ball

rolls around the cushion, considering only trajectories in which it bounces to and fro. We assume that it moves in a straight line between bounces, and that at each bounce the line of incidence and the line of departure make equal angles with the normal to the boundary at the point of impact.

We may parametrise the set of states at which the ball leaves the cushion by two parameters q and θ, where q is the point in Γ at which the ball leaves, and θ is the angle between the line it departs on and the tangent to Γ, in the counterclockwise direction (as shown in Figure 1.3). Thus the state space is $X = \Gamma \times (0,\pi)$, and the dynamical step $\phi : X \to X$ is the map that takes (q_0,θ_0) to (q_1,θ_1), where (q_1,θ_1) parametrises the state that results one bounce after the state parametrised by (q_0,θ_0).

If we denote by τ the bijection of X defined by $\tau(q,\theta) = (q,\pi-\theta)$, which reverses the direction of travel, then we have $\tau\circ\tau = \mathbb{1}$, and

$$(\phi\circ\tau\circ\phi)(q_0,\theta_0) = (\phi\circ\tau)(q_1,\theta_1) = \phi(q_1,\pi-\theta_1) = (q_0,\pi-\theta_0) = \tau(q_0,\theta_0)$$

so that, again, equation (1.2) holds.

Henceforth we omit the symbol $\circ$ from equations such as $\tau\circ\phi\circ\tau = \phi^{-1}$, unless the omission is likely to cause confusion.

1.1.4 Significance of the equation $\tau\phi\tau = \phi^{-1}$

Equation (1.2) says that the dynamical step ϕ is conjugate to its own inverse by the involutive map τ. Birkhoff was probably the first to point out the significance of this equation for a dynamical system [31, page 311]. It implies that the dynamics of the system are essentially the same as the dynamics of the inverse system. This has strong consequences. For instance, if a periodic point is fixed by the involution, then it cannot be attracting. Billiards always has periodic points of all orders [31, page 328ff]. This is a consequence of the famous *Last Geometric Theorem of Poincaré*, conjectured by Poincaré and proven eventually by Birkhoff. Period-two points correspond to orbits that bounce back and forth between two boundary points, and obviously each of the two states is fixed by the map τ that reverses the direction of travel. Thus these orbits are necessarily neither attracting nor repelling.

As we shall see in Chapter 3, the very same equation comes up when one considers the symmetry group of a regular polygon, a *dihedral group*, and we shall meet the equation later in many other situations in geometry, algebra, and analysis.

1.2 Origins in finite group theory

Reversibility is significant in the theory of finite groups because of its connections with representation theory. The next theorem, which is proven in Chapter 3, is the primordial result about reversibility in finite group theory.

Theorem 1.2 *An element g of a finite group G is conjugate to g^{-1} if and only if $\chi(g)$ is real for each complex character χ of G.* □

All the representations we consider in this section are complex representations.

It is because of this theorem that elements of finite groups that are conjugate to their own inverses are called *real* elements. However, we prefer to use the term *reversible* instead of *real* because most of the groups we consider later are infinite groups, for which there is no such characterisation of reversible elements using real characters.

In Chapter 3 we also prove that the number of conjugacy classes of reversible elements in a finite group is equal to the number of real-valued irreducible characters.

Theorem 1.2 tells us that we can identify the reversible elements of a finite group by studying the group's character table. We can obtain another result of the same type, using a similar (if slightly harder) proof.

Theorem 1.3 *An element g of a finite group G is conjugate to g^m for each integer m coprime to $|G|$ if and only if $\chi(g)$ is rational for each character χ of G.* □

Theorems 1.2 and 1.3 indicate that the structure of a finite group is closely related to the values taken by the group's characters. To investigate this relationship more thoroughly, group theorists use the *Schur index* of a character, which we describe briefly.

Let χ be an irreducible complex character of G. Given a subfield k of $\mathbb{C}$, we say that χ can be *realised over* k if there is an irreducible representation ϕ that has character χ such that the matrix entries of $\phi(g)$ lie in k, for every element g of G. Given a field F such that $\mathbb{Q} \subset F \subset \mathbb{C}$, we define $F(\chi)$ be the smallest subfield of $\mathbb{C}$ that contains F and all the values $\chi(g)$, for $g \in G$. The *Schur index of* χ *over* F, denoted $m_F(\chi)$, is the smallest degree of an extension of $F(\chi)$ over which χ can be realised. If $F = \mathbb{R}$, then we call $m_{\mathbb{R}}(\chi)$ the *Schur index of* χ.

The only possible values of $m_{\mathbb{R}}(\chi)$ are 1 and 2 because the only algebraic extensions of $\mathbb{R}$ are $\mathbb{R}$ and $\mathbb{C}$. The next theorem, the Brauer–Speiser theorem [85], invests this trivial observation with a great deal of value.

Theorem 1.4 *If χ is a real-valued irreducible character of a finite group G, then $m_{\mathbb{Q}}(\chi)$ is either 1 or 2.* □

In particular, if $m_{\mathbb{R}}(\chi)$ is 2, then because $\mathbb{Q}(\chi) \subset \mathbb{R}(\chi)$ it follows that $m_{\mathbb{Q}}(\chi)$ is also 2.

An important tool for calculating $m_{\mathbb{R}}(\chi)$ is the *Frobenius–Schur indicator* $v_2(\chi)$, which is defined by

$$v_2(\chi) = \frac{1}{|G|}\sum_{g\in G}\chi(g^2).$$

The connection between the Frobenius–Schur indicator and the Schur index is explained by the following result [138, page 58].

Theorem 1.5 *Given an irreducible character χ, the Frobenius–Schur indicator $v_2(\chi)$ is either* 0, 1 *or* 2. *Furthermore,*

$$v_2(\chi) = \begin{cases} 0, & \text{if } \chi \text{ is not real-valued,} \\ 1, & \text{if } \chi \text{ is realised over } \mathbb{R}, \\ -1, & \text{if } \chi \text{ is real-valued, but cannot be realised over } \mathbb{R}. \end{cases}$$

□

Clearly, if $v_2(\chi)$ is 0 or 1, then $m_{\mathbb{R}}(\chi) = 1$, and if $v_2(\chi) = -1$, then $m_{\mathbb{R}}(\chi) = 2$.

Theorem 1.5 has an important application in counting square roots of elements. It is proven in [138, page 49] and [140, Corollary 23.17] that the number of square roots in G of an element g is given by

$$\sum v_2(\chi)\chi(g),$$

where the sum is taken over all irreducible characters of G. This formula can be used to help prove the Brauer–Fowler theorem about centralisers of involutions in finite simple groups (see [140, Chapter 23]), which is a pivotal result in the classification of the finite simple groups.

The standard proof of Theorem 1.5 throws up an interesting subsidiary result [138, Theorem 4.14] (in which we denote the transpose of a matrix A by A^t).

Theorem 1.6 *Suppose that ϕ is an irreducible representation with a real-valued character χ. Then there exists a nonzero square matrix M such that*

$$\phi(g)^t M\phi(g) = M$$

for all elements g of G. Furthermore, for any such matrix M, we have $M^t = v_2(\chi)M$. □

It follows from Theorem 1.6 that the image of the representation $\phi : G \to \mathrm{GL}(V)$ lies inside the isometry group of the bilinear form

$$\beta : V\times V \to \mathbb{C}, \quad (u,v)\mapsto u^t Mv.$$

If $\nu_2(\chi) = 1$, then $M^t = M$, and β is a symmetric bilinear form; in this case ϕ is called an *orthogonal representation*. If $\nu_2(\chi) = -1$, then $M^t = -M$, and β is a skew-symmetric bilinear form; in this case ϕ is called a *symplectic representation*.

It is helpful to summarise parts of Theorems 1.5 and 1.6 in the following corollary.

Corollary 1.7 *Let χ be an irreducible character. The following are equivalent:*

(i) $m_{\mathbb{R}}(\chi) = 2$
(ii) $\nu_2(\chi) = -1$
(iii) *χ is real-valued and is realised by a symplectic representation.* □

Brauer [36, Problem 14] asked for a group theoretic description of the number of irreducible characters of a finite group with Schur index 1. This question has been answered by Gow [119] when the Sylow 2-subgroups of G are nontrivial and cyclic, and Gow's answer is given in terms of the number of reversible conjugacy classes of G.

A slightly easier question than Brauer's asks for a characterisation of those groups G such that *all* real-valued irreducible characters χ have Schur index 1. A partial answer to this was given by Gow in [119, Corollary 1]. To state Gow's result, we recall that a reversible element of a group is *strongly reversible* if it is conjugate to its inverse by an involution.

Theorem 1.8 *Let G be a finite group whose Sylow 2-subgroup is abelian. Then all real-valued irreducible characters χ of G have Schur index 1 if and only if all reversible elements of G are strongly reversible.*

See [118, 120] for related work of Gow.

Gow notes that often (but not always) the existence of real-valued irreducible characters with Schur index 2 is accompanied by the existence of reversible elements that are not strongly reversible. A conjecture in [146], which the authors of [146] attribute to Tiep, makes this more precise. To understand the conjecture, remember that the Schur index $m_{\mathbb{R}}(\chi) = 1$ if and only if the Frobenius–Schur indicator $\nu_2(\chi)$ is 0 or 1. Finite groups for which all irreducible characters χ satisfy $\nu_2(\chi) = 1$ are called *totally orthogonal* because, as we have seen, all their representations are orthogonal.

Conjecture 1.9 *A finite simple group is totally orthogonal if and only if all its elements are strongly reversible.*

It is therefore of interest to determine all those finite simple groups whose elements are all strongly reversible. This very question appeared in the famous Kourovka notebook [178], as Problem 14.82, posed by Sozutov (but described as a "well-known problem"). In 2005, Tiep and Zalesski [231] classified all the simple (and quasi-simple) finite groups in which all elements are reversible. Results of [22, 81, 93, 121, 122, 153, 154, 202, 233] imply that each of these finite simple groups that consists entirely of reversible elements in fact consists entirely of strongly-reversible elements (therefore Problem 14.82 is solved). To prove Tiep's conjecture, then, one must calculate $\nu_2(\chi)$ for all irreducible characters χ of each of the finite simple groups.

Recently, Kaur and Kulsherstha [146] have shown that the conjecture is false if the finite groups are no longer required to be simple. They construct infinite families of special 2-groups that are totally orthogonal but do not consist entirely of strongly-reversible elements, and they also construct infinite families of special 2-groups that consist entirely of strongly-reversible elements but are not totally orthogonal.

1.3 Origins in hidden dynamics

To further motivate our formal study of reversibility, we briefly describe some examples in which the dynamics of a reversible map play a key role in resolving a problem with no apparent dynamic connection; this phenomenon is known as *hidden dynamics*. Our first example is elementary, our second example is from complex analysis, and our third example is from approximation theory.

1.3.1 Small fibres

A map $f : X \to Y$ is said to have *small fibres* if $f^{-1}(f(x))$ has cardinality at most two, for each x in X. Whenever f has small fibres, we may define an associated involution $\tau : X \to X$ by requiring that

$$f^{-1}(f(x)) = \{x, \tau(x)\}, \quad \text{for } x \text{ in } X.$$

In other words, τ swaps the preimages of each point, if there are two, and fixes the preimage, if there is only one.

Many interesting involutions arise in this way. (In fact, all involutions arise in this way: given an involution τ on a set X, we may define Y as the space of orbits of τ and $f : X \to Y$ as the quotient map.) For example, the quadratic map

$$f : \mathbb{C} \to \mathbb{C}, \quad z \mapsto z^2 + bz,$$

induces the involution $z \mapsto -z-b$. The cubic map

$$f:\mathbb{C}\to\mathbb{C}, \quad z\mapsto z^3+z^2,$$

has three preimages for most points, but if restricted to a small enough neighbourhood of the origin, it has at most two-point fibres, and the induced involution τ is actually holomorphic on a neighbourhood of 0, with

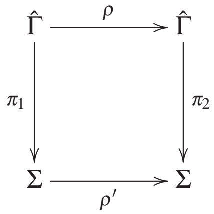
$$\tau(z) = -z - z^2 + \cdots .$$

It may happen that a problem leads us to two maps $f_1 : X \to Y_1$ and $f_2 : X \to Y_2$, each having small fibres. From the induced involutions τ_1 and τ_2 we can define $\phi = \tau_1\tau_2$, a reversible bijective map of X. This scenario arises in a significant number of cases, and often the dynamics of ϕ prove useful.

1.3.2 Two-valued reflections

Webster [240, 241, 242] exploited the small fibres idea in connection with the concept of two-valued reflections. Two-valued reflections occur when the familiar local antiholomorphic reflection across a real-analytic curve γ on a Riemann surface Σ happens to have precisely two global extensions. This happens, for instance, for reflection across an ellipse in the sphere, and also for certain particular quartic lemniscates.

The abstract situation involves a compact Riemann surface $\hat{\Gamma}$ (related to the complexification Γ of γ) and maps π_1, π_2, ρ, and ρ', where π_1 and π_2 are each two-fold branched covers of Σ, ρ is an antiholomorphic involution of $\hat{\Gamma}$, and ρ' is an antiholomorphic involution of Σ, such that the following diagram commutes.



When Σ is the Riemann sphere $\hat{\mathbb{C}}$, it turns out that the existence of both π_1 and π_2 implies (by the Riemann–Roch theorem) that $\hat{\Gamma}$ is the Riemann sphere or a torus. Webster considered these cases, and discovered which Γ correspond to various curves, by an argument that produced explicit parametrisations of Γ and formulas for the reflections. The point is that these formulas are found by a natural process of discovery; we do not need a stroke of genius to find them; nothing has to be pulled out of the air.