

## Contents

Preface	<i>page</i> xi
Acknowledgements	xiii
Introduction	xv
(a) What is a moduli space?	xv
(b) Algebraic varieties and quotients of algebraic varieties	xvi
(c) Moduli of bundles on a curve	xix
<b>1 Invariants and moduli</b>	1
1.1 A parameter space for plane conics	1
1.2 Invariants of groups	9
(a) Hilbert series	9
(b) Molien's formula	13
(c) Polyhedral groups	15
1.3 Classical binary invariants	19
(a) Resultants and discriminants	19
(b) Binary quartics	26
1.4 Plane curves	32
(a) Affine plane curves	32
(b) Projective plane curves	35
1.5 Period parallelograms and cubic curves	41
(a) Invariants of a lattice	41
(b) The Weierstrass $\wp$ function	44
(c) The $\wp$ function and cubic curves	47
<b>2 Rings and polynomials</b>	51
2.1 Hilbert's Basis Theorem	51
2.2 Unique factorisation rings	55
2.3 Finitely generated rings	58

vi	<i>Contents</i>	
2.4	Valuation rings	61
	(a) Power series rings	61
	(b) Valuation rings	63
2.5	A diversion: rings of invariants which are not finitely generated	68
	(a) Graded rings	69
	(b) Nagata's trick	70
	(c) An application of Liouville's Theorem	73
<b>3</b>	<b>Algebraic varieties</b>	77
3.1	Affine varieties	78
	(a) Affine space	78
	(b) The spectrum	81
	(c) Some important notions	86
	<i>Morphisms</i>	86
	<i>Products</i>	87
	<i>General spectra and nilpotents</i>	88
	<i>Dominant morphisms</i>	89
	<i>Open immersions</i>	90
	<i>Local properties</i>	91
3.2	Algebraic varieties	91
	(a) Gluing affine varieties	91
	(b) Projective varieties	95
3.3	Functors and algebraic groups	98
	(a) A variety as a functor from algebras to sets	98
	(b) Algebraic groups	100
3.4	Completeness and toric varieties	103
	(a) Complete varieties	103
	(b) Toric varieties	107
	(c) Approximation of valuations	111
<b>4</b>	<b>Algebraic groups and rings of invariants</b>	116
4.1	Representations of algebraic groups	117
4.2	Algebraic groups and their Lie spaces	122
	(a) Local distributions	122
	(b) The distribution algebra	124
	(c) The Casimir operator	128
4.3	Hilbert's Theorem	130
	(a) Linear reductivity	130
	(b) Finite generation	135
4.4	The Cayley-Sylvester Counting Theorem	137
	(a) $SL(2)$	137
	(b) The dimension formula for $SL(2)$	140

<i>Contents</i>		vii
	(c) A digression: Weyl measure	142
	(d) The Cayley-Sylvester Formula	143
	(e) Some computational examples	148
	4.5 Geometric reductivity of $SL(2)$	152
<b>5</b>	<b>The construction of quotient varieties</b>	<b>158</b>
	5.1 Affine quotients	159
	(a) Separation of orbits	159
	(b) Surjectivity of the affine quotient map	163
	(c) Stability	165
	5.2 Classical invariants and the moduli of smooth hypersurfaces in $\mathbb{P}^n$	167
	(a) Classical invariants and discriminants	167
	(b) Stability of smooth hypersurfaces	171
	(c) A moduli space for hypersurfaces in $\mathbb{P}^n$	174
	(d) Nullforms and the projective quotient map	175
<b>6</b>	<b>The projective quotient</b>	<b>181</b>
	6.1 Extending the idea of a quotient: from values to ratios	182
	(a) The projective spectrum	186
	(b) The Proj quotient	189
	(c) The Proj quotient by a $GL(n)$ action of ray type	195
	6.2 Linearisation and Proj quotients	197
	6.3 Moving quotients	201
	(a) Flops	201
	(b) Toric varieties as quotient varieties	205
	(c) Moment maps	208
<b>7</b>	<b>The numerical criterion and some applications</b>	<b>211</b>
	7.1 The numerical criterion	212
	(a) 1-parameter subgroups	212
	(b) The proof	213
	7.2 Examples and applications	219
	(a) Stability of projective hypersurfaces	219
	(b) Cubic surfaces	224
	(c) Finite point sets in projective space	230
<b>8</b>	<b>Grassmannians and vector bundles</b>	<b>234</b>
	8.1 Grassmannians as quotient varieties	235
	(a) Hilbert series	237
	(b) Standard monomials and the ring of invariants	239
	(c) Young tableaux and the Plücker relations	241
	(d) Grassmannians as projective varieties	245
	(e) A digression: the degree of the Grassmannian	247

Cambridge University Press

978-1-107-40636-0 - An Introduction to Invariants and Moduli

Shigeru Mukai

Table of Contents

[More information](#)

viii	<i>Contents</i>	
8.2	Modules over a ring	251
	(a) Localisation	251
	(b) Local versus global	254
	(c) Free modules	257
	(d) Tensor products and flat modules	259
8.3	Locally free modules and flatness	262
	(a) Locally free modules	262
	(b) Exact sequences and flatness	264
8.4	The Picard group	268
	(a) Algebraic number fields	268
	(b) Two quadratic examples	271
8.5	Vector bundles	276
	(a) Elementary sheaves of modules	277
	(b) Line bundles and vector bundles	279
	(c) The Grassmann functor	282
	(d) The tangent space of the functor	284
<b>9</b>	<b>Curves and their Jacobians</b>	287
9.1	Riemann's inequality for an algebraic curve	288
	(a) Prologue: gap values and the genus	290
	(b) Divisors and the genus	292
	(c) Divisor classes and vanishing index of speciality	294
9.2	Cohomology spaces and the genus	297
	(a) Cousin's problem	297
	(b) Finiteness of the genus	301
	(c) Line bundles and their cohomology	304
	(d) Generation by global sections	307
9.3	Nonsingularity of quotient spaces	309
	(a) Differentials and differential modules	309
	(b) Nonsingularity	311
	(c) Free closed orbits	313
9.4	An algebraic variety with the Picard group as its set of points	316
	(a) Some preliminaries	316
	(b) The construction	319
	(c) Tangent spaces and smoothness	323
9.5	Duality	327
	(a) Dualising line bundles	327
	(b) The canonical line bundle	330
	(c) De Rham cohomology	332

<i>Contents</i>		ix
9.6	The Jacobian as a complex manifold	334
	(a) Compact Riemann surfaces	335
	(b) The comparison theorem and the Jacobian	337
	(c) Abel's Theorem	342
<b>10</b>	<b>Stable vector bundles on curves</b>	<b>348</b>
10.1	Some general theory	349
	(a) Subbundles and quotient bundles	350
	(b) The Riemann-Roch formula	352
	(c) Indecomposable bundles and stable bundles	355
	(d) Grothendieck's Theorem	359
	(e) Extensions of vector bundles	361
10.2	Rank 2 vector bundles	365
	(a) Maximal line subbundles	365
	(b) Nonstable vector bundles	366
	(c) Vector bundles on an elliptic curve	369
10.3	Stable bundles and Pfaffian semiinvariants	371
	(a) Skew-symmetric matrices and Pfaffians	371
	<i>Even skew-symmetric matrices</i>	372
	<i>Odd skew-symmetric matrices</i>	374
	<i>Skew-symmetric matrices of rank 2</i>	375
	(b) Gieseker points	376
	(c) Semistability of Gieseker points	379
10.4	An algebraic variety with $SU_C(2, L)$ as its set of points	386
	(a) Tangent vectors and smoothness	387
	(b) Proof of Theorem 10.1	391
	(c) Remarks on higher rank vector bundles	394
<b>11</b>	<b>Moduli functors</b>	<b>398</b>
11.1	The Picard functor	400
	(a) Fine moduli and coarse moduli	400
	(b) Cohomology modules and direct images	403
	(c) Families of line bundles and the Picard functor	407
	(d) Poincaré line bundles	410
11.2	The moduli functor for vector bundles	413
	(a) Rank 2 vector bundles of odd degree	416
	(b) Irreducibility and rationality	418
	(c) Rank 2 vector bundles of even degree	419
11.3	Examples	422
	(a) The Jacobian of a plane quartic	422
	(b) The affine Jacobian of a spectral curve	424

Cambridge University Press

978-1-107-40636-0 - An Introduction to Invariants and Moduli

Shigeru Mukai

Table of Contents

[More information](#)

x

*Contents*

	(c) The Jacobian of a curve of genus 1	425
	(d) Vector bundles on a spectral curve	431
	(e) Vector bundles on a curve of genus 2	433
<b>12</b>	<b>Intersection numbers and the Verlinde formula</b>	437
12.1	Sums of inverse powers of trigonometric functions	439
	(a) Sine sums	439
	(b) Variations	441
	(c) Tangent numbers and secant numbers	444
12.2	Riemann-Roch theorems	447
	(a) Some preliminaries	448
	(b) Hirzebruch-Riemann-Roch	450
	(c) Grothendieck-Riemann-Roch for curves	453
	(d) Riemann-Roch with involution	455
12.3	The standard line bundle and the Mumford relations	457
	(a) The standard line bundle	457
	(b) The Newstead classes	461
	(c) The Mumford relations	463
12.4	From the Mumford relations to the Verlinde formula	465
	(a) Warming up: secant rings	466
	(b) The proof of formulae (12.2) and (12.4)	470
12.5	An excursion: the Verlinde formula for quasiparabolic bundles	476
	(a) Quasiparabolic vector bundles	476
	(b) A proof of (12.6) using Riemann-Roch and the Mumford relations	480
	(c) Birational geometry	483
	Bibliography	487
	Index	495