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SHIGERU MUKAI

Translated by W. M. Oxbury

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Preface

The aim of this book is to provide a concise introduction to algebraic geometry and to algebraic moduli theory. In so doing, I have tried to explain some of the fundamental contributions of Cayley, Hilbert, Nagata, Grothendieck and Mumford, as well as some important recent developments in moduli theory, keeping the proofs as elementary as possible. For this purpose we work throughout in the category of algebraic varieties and elementary sheaves (which are simply order-reversing maps) instead of schemes and sheaves (which are functors). Instead of taking GIT (Geometric Invariant Theory) quotients of projective varieties by $PGL(N)$, we take, by way of a shortcut, Proj quotients of affine algebraic varieties by the general linear group $GL(N)$. In constructing the moduli of vector bundles on an algebraic curve, Grothendieck's Quot scheme is replaced by a certain explicit affine variety consisting of matrices with polynomial entries. In this book we do not treat the very important analytic viewpoint represented by the Kodaira-Spencer and Hodge theories, although it is treated, for example, in Ueno [113], which was in fact a companion volume to this book when published in Japanese.

The plan of the first half of this book (Chapters 1–5 and 7) originated from notes taken by T. Hayakawa in a graduate lecture course given by the author in Nagoya University in 1985, which in turn were based on the works of Hilbert [20] and Mumford et al. [30]. Some additions and modifications have been made to those lectures, as follows.

(1) I have included chapters on ring theory and algebraic varieties accessible also to undergraduate students. A strong motivation for doing this, in fact, was the desire to collect in one place the early series of fundamental results of Hilbert that includes the Basis Theorem and the Nullstellensatz.
(2) For the proof of linear reductivity (or complete reductivity), Cayley's Ω-process used by Hilbert is quite concrete and requires little background

knowledge. However, in view of the importance of algebraic group representations I have used instead a proof using Casimir operators. The key to the proof is an invariant bilinear form on the Lie space. The uniqueness property used in the Japanese edition was replaced by the positive definiteness in this edition.

(3) I have included the Cayley-Sylvester formula in order to compute explicitly the Hilbert series of the classical binary invariant ring since I believe both tradition and computation are important. I should add that this and Section 4.5 are directly influenced by Springer [8].

Both (2) and (3) took shape in a lecture course given by the author at Warwick University in the winter of 1998.

(4) I have included the result of Nagata [11], [12] that, even for an algebraic group acting on a polynomial ring, the ring of invariants need not be finitely generated.
(5) Chapter 1 contains various introductory topics adapted from lectures given in the spring of 1998 at Nagoya and Kobe Universities.

The second half of the book was newly written in 1998–2000 with two main purposes: first, an elementary invariant-theoretic construction of moduli spaces including Jacobians and, second, a self-contained proof of the Verlinde formula for $SL(2)$. For the first I make use of Gieseker matrices. Originally this idea was invented by Gieseker [72] to measure the stability of the action of $PGL(N)$ on the Quot scheme. But in this book moduli spaces of bundles are constructed by taking quotients of a variety of Gieseker matrices themselves by the general linear group. This construction turns out to be useful even in the case of Jacobians. For the Verlinde formula, I have chosen Zagier's proof [115] among three known algebraic geometric proofs. However, Thaddeus's proof [112] uses some interesting birational geometry, and I give a very brief explanation of this for the case of rank 2 parabolic bundles on a pointed projective line.

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Shigeru Mukai
Kyoto, June 2002

Introduction

(a) What is a moduli space?

A *moduli space* is a manifold, or variety, which parametrises some class of geometric objects. The j-invariant classifying elliptic curves up to isomorphism and the Jacobian variety of an algebraic curve are typical examples. In a broader sense, one could include as another classical example the classifying space of a Lie group. In modern mathematics the idea of moduli is in a state of continual evolution and has an ever-widening sphere of influence. For example:

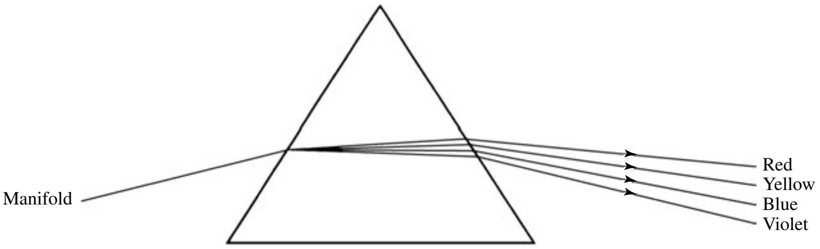
- By defining a suitable height function on the moduli space of principally polarised abelian varieties it was possible to resolve the Shafarevich conjectures on the finiteness of abelian varieties (Faltings 1983).
- The moduli space of Mazur classes of 2-dimensional representations of an absolute Galois group is the spectrum of a Hecke algebra.

The application of these results to resolve such number-theoretic questions as Mordell's Conjecture and Fermat's Last Theorem are memorable achievements of recent years. Turning to geometry:

- Via Donaldson invariants, defined as the intersection numbers in the moduli space of instanton connections, one can show that there exist homeomorphic smooth 4-manifolds that are not diffeomorphic.

Indeed, Donaldson's work became a prototype for subsequent research in this area.

Here's an anology. When natural light passes through a prism it separates into various colours. In a similar way, one can try to elucidate the hidden properties of an algebraic variety. One can think of the moduli spaces naturally associated to the variety (the Jacobian of a complex curve, the space of instantons on a complex surface) as playing just such a role of 'nature's hidden colours'.



The aim of this book is to explain, with the help of some concrete examples, the basic ideas of moduli theory as they have developed alongside algebraic geometry – in fact, from long before the modern viewpoint sketched above. In particular, I want to give a succinct introduction to the widely applicable methods for constructing moduli spaces known as *geometric invariant theory*.

If a moduli problem can be expressed in terms of algebraic geometry then in many cases it can be reduced to the problem of constructing a quotient of a suitable algebraic variety by an action of a group such as the general linear group $GL(m)$. From the viewpoint of moduli theory this variety will typically be a Hilbert scheme parametrising subschemes of a variety or a Quot scheme parametrising coherent sheaves. From a group-theoretic point of view it may be a finite-dimensional linear representation regarded as an affine variety or a subvariety of such. To decide what a solution to the quotient problem should mean, however, forces one to rethink some rather basic questions: What *is* an algebraic variety? What does it mean to take a quotient of a variety? In this sense the quotient problem, present from the birth and throughout the development of algebraic geometry, is even today sadly lacking an ideal formulation. And as one sees in the above examples, the ‘moduli problem’ is not determined in itself but depends on the methods and goals of the mathematical area in which it arises. In some cases elementary considerations are sufficient to address the problem, while in others much more care is required. Maybe one cannot do without a projective variety as quotient; maybe a stack or algebraic space is enough. In this book we will construct moduli spaces as projective algebraic varieties.

(b) Algebraic varieties and quotients of algebraic varieties

An algebraic curve is a rather sophisticated geometric object which, viewed on the one hand as a Riemann surface, or on the other as an algebraic function field in one variable, combines analysis and algebra. The theory of meromorphic functions and abelian differentials on compact Riemann surfaces, developed by Abel, Riemann and others in the nineteenth century, was, through the efforts of many later mathematicians, deepened and sublimated to an 'algebraic function theory'. The higher dimensional development of this theory has exerted a profound influence on the mathematics of the twentieth century. It goes by the general name of 'the study of algebraic varieties'. The data of an algebraic variety incorporate in a natural way that of real differentiable manifolds, of complex manifolds, or again of an algebraic function field in several variables. (A field K is called an algebraic function field in n variables over a base field k if it is a finitely generated extension of k of transcendence degree n.) Indeed, any algebraic variety may be defined by patching together (the spectra of) some finitely generated subrings $R_1, \ldots, R_N$ of a function field K. This will be explained in Chapter 3.

This ring-theoretic approach, from the viewpoint of varieties as given by systems of algebraic equations, is very natural; however, the moduli problem, that is, the problem of constructing quotients of varieties by group actions, becomes rather hard. When an algebraic group G acts on an affine variety, how does one construct a quotient variety? (An *algebraic group* is an algebraic variety with a group structure, just as a Lie group is a smooth manifold which has a compatible group structure.) It turns out that the usual quotient topology, and the differentiable structure on the quotient space of a Lie group by a Lie subgroup, fail to work well in this setting. Clearly they are not sufficient if they fail to capture the function field, together with its appropriate class of subrings, of the desired quotient variety. The correct candidates for *these* are surprisingly simple, namely, the subfield of G-invariants in the original function field K, and the subrings of G-invariants in the integral domains $R \subset K$ (see Chapter 5). However, in proceeding one is hindered by the following questions.

(1) Is the subring of invariants R^G of a finitely generated ring R again finitely generated?
(2) Is the subfield of invariants $K^G \subset K$ equal to the field of fractions of R^G?
(3) Is $K^G \subset K$ even an algebraic function field? – that is, is K^G finitely generated over the base field k?
(4) Even if the previous questions can be answered positively and an algebraic variety constructed accordingly, does it follow that the points of this variety can be identified with the G-orbits of the original space?

In fact one can prove property (3) quite easily; the others, however, are not true in general. We shall see in Section 2.5 that there exist counterexamples to (1) even in the case of an algebraic group acting linearly on a polynomial ring. Question (2) will be discussed in Chapter 6.

So how should one approach this subject? Our aim in this book is to give a concrete construction of some basic moduli spaces as quotients of group actions, and in fact we will restrict ourselves exclusively to the general linear group $GL(m)$. For this case property (1) does indeed hold (Chapter 4), and also property (4) if we modify the question slightly. (See the introduction to Chapter 5.) A correspondence between G-orbits and points of the quotient is achieved provided we restrict, in the original variety, to the open set of *stable points* for the group action. Both of these facts depend on a representation-theoretic property of $GL(m)$ called linear reductivity.

After paving the way in Chapter 5 with the introduction of affine quotient varieties, we 'globalise' the construction in Chapter 6. Conceptually, this may be less transparent than the affine construction, but essentially it just replaces the affine spectrum of the invariant ring with the projective spectrum (Proj) of the semiinvariant ring. This 'global' quotient, which is a projective variety, we refer to as the *Proj quotient*, rather than 'projective quotient', in order to distinguish it from other constructions of the projective quotient variety that exist in the literature.

An excellent example of a Proj quotient (and indeed of a moduli space) is the Grassmannian. In fact, the Grassmannian is seldom considered in the context of moduli theory, and we discuss it here in Chapter 8. This variety is usually built by gluing together affine spaces, but here we construct it globally as the projective spectrum of a semiinvariant ring and observe that this is equivalent to the usual construction. For the Grassmannian $\mathbb{G}(2, n)$ we compute the Hilbert series of the homogeous coordinate ring. We use this to show that it is generated by the Plücker coordinates, and that the relations among these are generated by the Plücker relations.

In general, for a given moduli problem, one can only give an honest construction of a moduli space if one is able to determine explicitly the stable points of the group action. This requirement of the theory is met in Chapter 7 with the numerical criterion for stability and semistability of Hilbert and Mumford, which we apply to some geometrical examples from Chapter 5. Later in the book we construct moduli spaces for line bundles and vector bundles on an algebraic curve, which requires the notion of stability of a vector bundle. Historically, this was discovered by Mumford as an application of the numerical criterion, but in this book we do not make use of this, as we are able to work directly with the semiinvariants of our group actions. Another important application, which we do not touch on here, is to the construction of a compactification of the moduli space of curves as a projective variety.

(c) Moduli of bundles on a curve

In Chapter 9 algebraic curves make their entry. We first explain:

(1) what is the genus of a curve?
(2) Riemann's inequality and the vanishing of cohomology (or index of speciality); and
(3) the duality theorem.

In the second half of Chapter 9 we construct, as the projective spectrum of the semiinvariant ring of a suitable group action on an affine variety, an algebraic variety whose underlying set of points is the Picard group of a given curve, and we show that over the complex numbers this is nothing other than the classical Jacobian.

In Chapter 10 we extend some essential parts of the line bundle theory of the preceding chapter to higher rank vector bundles on a curve, and we then construct the moduli space of rank 2 vector bundles. This resembles the line bundle case, but with the difference that the notion of stability arises in a natural way. The moduli space of vector bundles, in fact, can be viewed as a Grassmannian over the function field of the curve, and one can roughly paraphrase Chapter 10 by saying that a moduli space is constructed as a projective variety by explicitly defining the Plücker coordinates of a semistable vector bundle. (See also Seshadri [77].) One advantage of this construction – although it has not been possible to say much about this in this book – is the consequence that, if the curve is defined over a field k, then the same is true, a priori, of the moduli space.

In Chapter 11 the results of Chapters 9 and 10 are reconsidered, in the following sense. Algebraic varieties have been found whose sets of points can be identified with the sets of equivalence classes of line bundles, or vector bundles, on the curve. However, to conclude that 'these varieties are the moduli spaces for line bundles, or vector bundles' is not a very rigorous statement. More mathematical would be, first, to give some clean definition of 'moduli' and 'moduli space', and then to prove that the varieties we have obtained are moduli spaces in the sense of this definition. One answer to this problem is furnished by the notions of representability of a functor and of coarse moduli. These are explained in Chapter 11, and the quotient varieties previously constructed are shown to be moduli spaces in this sense. Again, this point of view becomes especially important when one is interested in the field over which the moduli space is defined. This is not a topic which it has been possible to treat in this book, although we do give one concrete example at the end of the chapter, namely, the Jacobian of an elliptic curve.

In the final chapter we give a treatment of the Verlinde formulae for rank 2 vector bundles. Originally, these arose as a general-dimension formula for objects that are somewhat unfamiliar in geometry, the spaces of conformal blocks from 2-dimensional quantum field theory. (See Ueno [113].) In our context, however, they appear as elegant and precise formulae for the Hilbert polynomials for the semiinvariant rings used to construct the moduli of vector bundles. Various proofs are known, but the one presented here (for odd degree bundles) is that of Zagier [115], making use of the formulae for the intersection numbers in the moduli space of Thaddeus [111]. On the way, we observe a curious formal similarity between the cohomology ring of the moduli space and that of the Grassmannian $\mathbb{G}(2, n)$.

Convention: Although it will often be unnecessary, we shall assume throughout the book that the field k is algebraically closed and of characteristic zero.