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Invariants and moduli

This chapter explores some examples of parameter spaces which can be constructed by elementary means and with little previous knowledge as an introduction to the general theory developed from Chapter 3 onwards. To begin, we consider equivalence classes of plane conics under Euclidean transformations and use invariants to construct a parameter space which essentially corresponds to the eccentricity of a conic.

This example already illustrates several essential features of the construction of moduli spaces. In addition we shall look carefully at some cases of finite group actions, and in particular at the question of how to determine the ring of invariants, the fundamental tool of the theory. We prove Molien's Formula, which gives the Hilbert series for the ring of invariants when a finite group acts linearly on a polynomial ring.

In Section 1.3, as an example of an action of an algebraic group, we use classical invariants to construct a parameter space for $GL(2)$-orbits of binary quartics.

In Section 1.4 we review plane curves as examples of algebraic varieties. A plane curve without singularities is a Riemann surface, and in the particular case of a plane cubic this can be seen explicitly by means of doubly periodic complex functions. This leads to another example of a quotient by a discrete group action, in this case parametrising lattices in the complex plane. The group here is the modular group $SL(2, \mathbb{Z})$ (neither finite nor connected), and the Eisenstein series are invariants. Among them one can use two, g_2 and g_3, to decide when two lattices are isomorphic.

1.1 A parameter space for plane conics

Consider the curve of degree 2 in the (real or complex) (x, y) plane

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0. \tag{1.1}$$

If the left-hand side factorises as a product of linear forms, then the curve is a union of two lines; otherwise we say that it is *nondegenerate* (Figure 1.1).

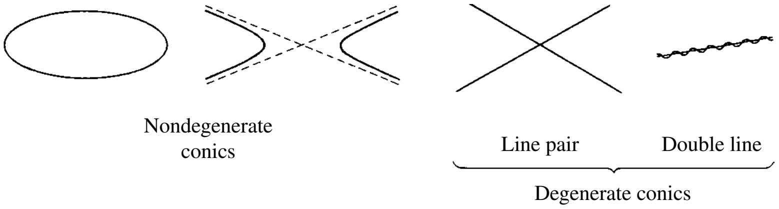

Figure 1.1

Let us consider the classification of such curves of degree 2, up to Euclidean transformations, from the point of view of their invariants. The Euclidean transformation group G contains the set of translations

$$x \mapsto x + l, \qquad y \mapsto y + m$$

as a normal subgroup and is generated by these and the rotations. Alternatively, G can be viewed as the group of matrices

$$X = \begin{pmatrix} p & q & l \\ -q & p & m \\ 0 & 0 & 1 \end{pmatrix}, \qquad p^2 + q^2 = 1. \tag{1.2}$$

Curves of degree 2 correspond to symmetric 3×3 matrices by writing the equation (1.1) as

$$(x, y, 1) \begin{pmatrix} a & b & d \\ b & c & e \\ d & e & f \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0,$$

and then under the Euclidean transformation (1.2) the symmetric matrix of the curve transforms by

$$\begin{pmatrix} a & b & d \\ b & c & e \\ d & e & f \end{pmatrix} \mapsto X^{\mathrm{t}} \begin{pmatrix} a & b & d \\ b & c & e \\ d & e & f \end{pmatrix} X.$$

In other words, the 6-dimensional vector space V of symmetric 3×3 matrices is a representation of the Euclidean transformation group G (see Section 1.21.10). Now, geometry studies properties which are invariant under groups of transformations, so let us look for invariants under this group action, in the form of polynomials $F(a, b, \ldots, f)$.

The transformation matrix (1.2) has determinant 1, and so the first invariant polynomial we encounter is

$$D = \det \begin{pmatrix} a & b & d \\ b & c & e \\ d & e & f \end{pmatrix}.$$

Here $D \neq 0$ exactly when the degree 2 curve is nondegenerate, and for this reason D is called the *discriminant* of the curve. Next we observe that the trace and determinant of the 2×2 submatrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ are also invariant; we will denote these by $T = a + c$ and $E = ac - b^2$. Moreover, any invariant polynomial can be (uniquely) expressed as a polynomial in D, T, E. In other words, the following is true.

Proposition 1.1. *The set of polynomials on V invariant under the action of G is a subring of $\mathbb{C}[a, b, c, d, e, f]$ and is generated by D, T, E. Moreover, these elements are algebraically independent; that is, the subring is $\mathbb{C}[D, T, E]$.* □

Proof. Let $G_0 \subset G$ be the translation subgroup, with quotient $G/G_0 \cong O(2)$, the rotation group of the plane. We claim that it is enough to show that the subring of polynomials invariant under G_0 is

$$\mathbb{C}[a, b, c, d, e, f]^{G_0} = \mathbb{C}[a, b, c, D]. \tag{1.3}$$

This is because the polynomials in $\mathbb{C}[a, b, c]$ invariant under the rotation group $O(2)$ are generated by the trace T and discriminant E.

We also claim that if we consider polymonials in a, b, c, d, e, f and $1/E$, then

$$\mathbb{C}\left[a, b, c, d, e, f, \frac{1}{E}\right]^{G_0} = \mathbb{C}\left[a, b, c, D, \frac{1}{E}\right]. \tag{1.4}$$

It is clear that this implies (1.3), and so we are reduced to proving (1.4). The point here is that the determinant D can be written

$$D = Ef + (2bde - ae^2 - cd^2),$$

so that

$$f = \frac{D + ae^2 + cd^2 - 2bde}{E},$$

and hence

$$\mathbb{C}\left[a, b, c, d, e, f, \frac{1}{E}\right] = \mathbb{C}\left[a, b, c, d, e, D, \frac{1}{E}\right].$$

So a polynomial F in this ring (that is, a polynomial in a, b, c, d, e, f with coefficients which may involve powers of $1/E$) which is invariant under G_0 has to satisfy

$$F(a, b, c, d + al + bm, e + bl + cm, D) = F(a, b, c, d, e, D)$$

for arbitrary translations (l, m). Taking $(l, m) = (-bt, at)$ shows that F cannot have terms involving e, while taking $(l, m) = (-ct, bt)$ shows that it cannot have terms involving d; so we have shown (1.4). □

Remark 1.2. One can see that Proposition 1.1 is consistent with a dimension count as follows. First, V has dimension 6. The Euclidean group G has dimension 3 (that is, Euclidean motions have 3 degrees of freedom). A general curve of degree 2 is preserved only by the finitely many elements of G (namely, 180° rotation about the centre and the trivial element), and hence we expect that 'the quotient V/G has dimension 3'. Thus we may think of the three invariants D, T, E as three 'coordinate functions on the quotient space'. □

The space of all curves of degree 2 is $V \cong \mathbb{C}^6$, but here we are only concerned with polynomials, viewed as functions, on this space. Viewed in this sense the space is called an affine space and denoted $\mathbb{A}^6$. (See Chapter 3.) We shall denote the subset corresponding to nondegenerate curves by $U \subset V$. This is an open set defined by the condition $D \neq 0$. The set of 'regular functions' on this open set is the set of rational functions on V whose denominator is a power of D, that is,

$$\mathbb{C}\left[a, b, c, d, e, f, \frac{1}{D}\right].$$

Up to now we have been thinking not in terms of curves but rather in terms of their defining equations of degree 2. In the following we shall want to think in terms of the curves themselves. Since two equations that differ only by a scalar multiple define the same curve, we need to consider functions that are invariant under the larger group $\widetilde{G}$ generated by G and the scalar matrices $X = rI$. The scalar matrix rI multiplies the three invariants D, E, T by r^6, r^4, r^2, respectively. It follows that the set

$$\mathbb{C}\left[a, b, c, d, e, f, \frac{1}{D}\right]^{\widetilde{G}}$$

of $\widetilde{G}$-invariant polynomial functions on U is generated by

$$A = \frac{E^3}{D^2}, \quad B = \frac{T^3}{D}, \quad C = \frac{ET}{D}.$$

Among these three expressions there is a relation

$$AB - C^3 = 0,$$

so that:

a moduli space for nondegenerate curves of degree 2 in the Euclidean plane is the affine surface in $\mathbb{A}^3$ defined by the equation $xz - y^3 = 0$.

(The origin is a singular point of this surface called a rational double point of type A_2.)

One can also see this easily in the following way. By acting on the defining equation (1.1) of a nondegenerate degree 2 curve with a scalar matrix rI for a suitable $r \in \mathbb{C}$ we can assume that $D(a, b, \ldots, f) = 1$. The set of curves normalised in this way is then an affine plane with coordinates T, E. Now, the ambiguity in choosing such a normalisation is just the action of ωI, where $\omega \in \mathbb{C}$ is an imaginary cube root of unity, and so the parameter space for nondegenerate degree 2 curves is the surface obtained by dividing out the (T, E) plane by the action of the cyclic group of order 3,

$$(T, E) \mapsto (\omega T, \omega^2 E).$$

The origin is a fixed point of this action, and so it becomes a quotient singularity in the parameter space.

Next, let us look at the situation over the real numbers $\mathbb{R}$. We note that here cube roots are uniquely determined, and so by taking that of the discriminant D of equation (1.1) we see that for real curves of degree 2 we can take as coordinates the numbers

$$\alpha = \frac{E}{\sqrt[3]{D^2}}, \qquad \beta = \frac{T}{\sqrt[3]{D}}.$$

In this way the curves are parametrised simply by the real (α, β) plane:

(i) Points in the (open) right-hand parabolic region $\beta^2 < 4\alpha$ and the (closed) 4th quadrant $\alpha \geq 0$, $\beta \leq 0$ do not correspond to any curves over the real numbers. (It is natural to refer to the union of these two sets as the 'imaginary region' of the (α, β) plane. See Figure 1.2.) The points of the parameter space are real, but the coefficients of the defining equation (1.1) always require imaginary complex numbers. For example, the origin (0, 0) corresponds to the curve

$$\sqrt{-1}(x^2 - y^2) + 2xy = 2x.$$

(ii) Points of the parabola $\beta^2 = 4\alpha$ in the 1st quadrant correspond to circles of radius $\sqrt{2/\beta}$.

(iii) Points of the open region $\beta^2 > 4\alpha > 0$ between the parabola and the β-axis parametrise ellipses.

(iv) Points of the positive β-axis $\alpha = 0$, $\beta > 0$ parametrise parabolas.

(v) Points in the left half-plane $\alpha < 0$ parametrise hyperbolas. Within this region, points along the negative α-axis parametrise rectangular hyperbolas (the graph of the reciprocal function), while points in the 2nd and 3rd quadrants correspond respectively to acute angled and obtuse angled hyperbolas.

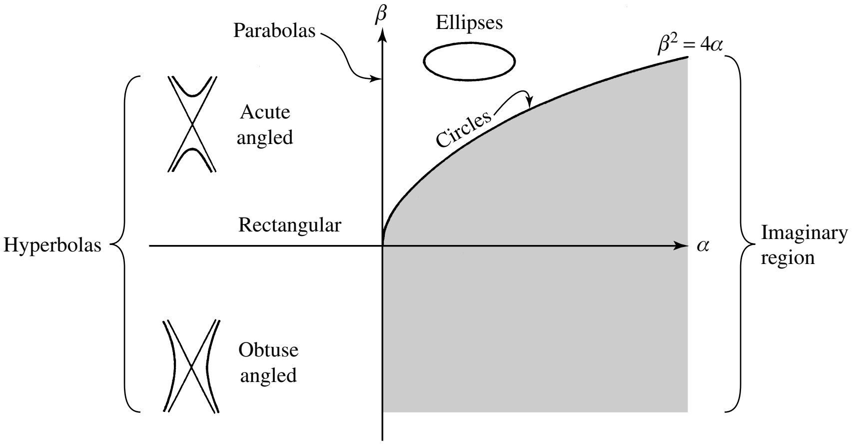

Figure 1.2: The parameter space of real curves of degree 2

Let us now follow a rotation of this figure in the positive direction about the origin.

Beginning with a circle (eccentricity $e = 0$), our curve grows into an ellipse through a parabolic phase ($e = 1$) before making a transition to a hyperbola. The angle between the asymptotes of this hyperbola is initially close to zero and gradually grows to 180°, at which point ($e = \infty$) the curve enters the imaginary region. After passing through this region it turns once again into a circle. (This is Kepler's Principle.)

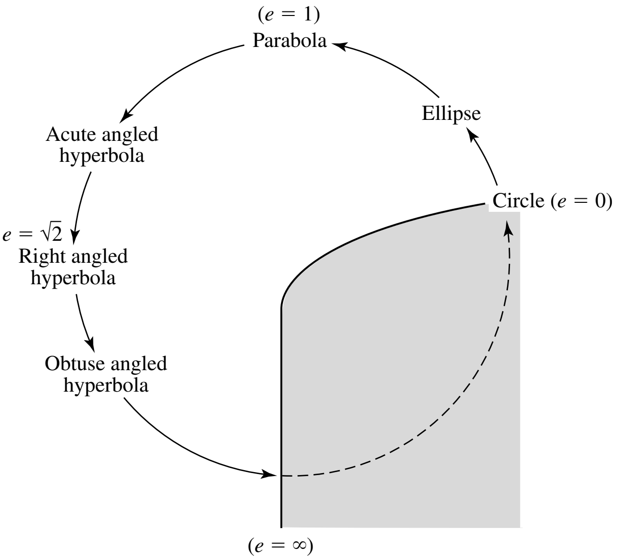

Figure 1.3: Transmigration of a conic

Remark 1.3. In the case of an ellipse, our curve has a (Euclidean invariant) area which is equal to $\pi/\sqrt{\alpha}$. In particular, this area increases as the curve approaches the β-axis, and one may think of a parabola, corresponding to a point on the axis, as having infinite area. Taking this point of view a step further, one may think of a hyperbola as having imaginary area. □

We have thus established a correspondence between real curves of degree 2 up to Euclidean transformations and points of the (α, β) plane. The group G does not have the best properties (it is not linearly reductive – this will be explained in Chapter 4), but nevertheless in this example we are lucky and every point of the (α, β) plane corresponds to some curve.

Plane curves of degree 2 are also called *conics*, as they are the curves obtained by taking plane cross sections of a circular cone (an observation which goes back to Apollonius and Pappus). From this point of view, the eccentricity e of the curve is determined by the angle of the plane (Figure 1.4).

To be precise, let ϕ be the angle between the axis of the cone and the circular base, and let ψ be the angle between the axis and the plane of the conic. If we now let

$$e = \frac{\sin\psi}{\sin\phi},$$

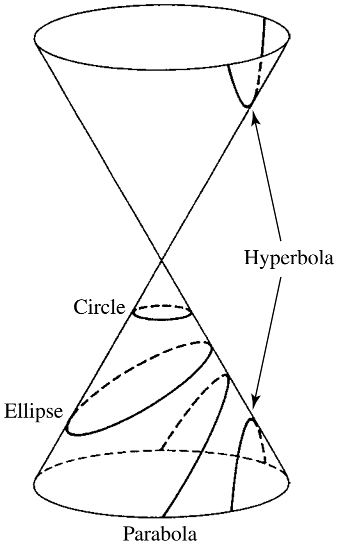

Figure 1.4: Plane sections of a cone

then for $e < 1$, $e = 1$ and $e > 1$, respectively, the conic section is an ellipse, a parabola or a hyperbola. As is well known, the eccentricity can also be expressed as

$$e = \frac{\text{distance from the focus}}{\text{distance to the directrix}}.$$

(For a curve with equation $(x/a)^2 \pm (y/b)^2 = 1$, where $a \leq b$, we find that $e = \sqrt{1 \mp (a/b)^2}$.) This is not an invariant polynomial function, but it satisfies an algebraic equation whose coefficients are invariants. Namely, it is the invariant multivalued function satisfying the quartic equation

$$(e^2 - 1) + \frac{1}{e^2 - 1} = 2 - \frac{T^2}{2E}.$$

Although e is properly speaking multivalued, we can take advantage of the fact that we are considering conics over the real numbers. In this case it is possible to choose a branch so that the function is single-valued for conics with real coefficients.

Suppose we extend the Euclidean transformation group to include also similarities (dilations and contractions). Transforming a conic by a scale factor k multiplies α by $\sqrt[3]{k^2}$ and multiplies β by $\sqrt[3]{k}$. So the 'moduli space' is now the

(α, β) plane, minus the origin, divided out by the action of scalars

$$(\alpha, \beta) \mapsto (\sqrt[3]{k^2}\alpha, \sqrt[3]{k}\beta).$$

In other words, it is a projective line (more precisely, the weighted projective line $\mathbb{P}(1:2)$; see Example 3.46 in Chapter 3). The one dimensional parameter that we obtain in this way is essentially the eccentricity e.

The aim of the first part of this book is to generalise the construction of this sort of parameter space to equivalence classes of polynomials in several variables under the action of the general linear group. In geometric language, our aim is to construct parameter spaces for equivalence classes of general-dimensional projective hypersurfaces with respect to projective transformations.

1.2 Invariants of groups

To say that a polynomial $f(x_1, \ldots, x_n)$ in n variables is an *invariant* with respect to an $n \times n$ matrix $A = (a_{ij})$ can have one of two meanings:

(i) f is invariant under the coordinate transformation determined by A. That is, it satisfies

$$f(Ax) := f\left(\sum_i a_{1i}x_i, \ldots, \sum_i a_{ni}x_i\right) = f(x). \tag{1.5}$$

(ii) f is invariant under the derivation

$$\mathcal{D}_A = \sum_{i,j} a_{ij}x_i \frac{\partial}{\partial x_j}$$

determined by A. In other words, it satisfies

$$\mathcal{D}_A f = \sum_{i,j} a_{ij}x_i \frac{\partial f}{\partial x_j} = 0. \tag{1.6}$$

In both cases, the invariant polynomials under some fixed set of matrices form a subring of $\mathbb{C}[x_1, \ldots, x_n]$. The idea of a Lie group and of a Lie algebra, respectively, arises in a natural way out of these two notions of invariants.

(a) Hilbert series

To begin, we review the first notion 1.2(i) of invariance. (The second will reappear in Chapter 4.) Given a set of nonsingular matrices $T \subset GL(n)$, we consider the set of all invariant polynomials

$$\{f \in \mathbb{C}[x_1, \ldots, x_n] \mid f(Ax) = f(x) \text{ for all } A \in T\}.$$

Clearly this is a subring of $\mathbb{C}[x_1, \ldots, x_n]$, called the *ring of invariants* of T. Notice that if $f(x)$ is an invariant under matrices A and B, then it is an invariant under the inverse A^{-1} and the product AB. It follows that in the definition of the ring of invariants we may assume without loss of generality that T is closed under taking products and inverses. This is just the definition of a group; moreover, in essence we have here the definition of a group representation.

Definition 1.4. Let $G \subset GL(n)$ be a subgroup. A polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ satisfying

$$f(Ax) = f(x) \text{ for all } A \in G$$

is called a G-*invariant*. □

We shall write $S = \mathbb{C}[x_1, \ldots, x_n]$ for the polynomial ring and S^G for the ring of invariants of G. Let us examine some cases in which G is a finite group.

Example 1.5. Let G be the symmetric group consisting of all $n \times n$ permutation matrices – that is, having a single 1 in each row and column, and 0 elsewhere. The invariants of G in $\mathbb{C}[x_1, \ldots, x_n]$ are just the symmetric polynomials. These form a subring which includes the n elementary symmetric polynomials

$$\begin{aligned} \sigma_1(x) &= \textstyle\sum_i x_i \\ \sigma_2(x) &= \textstyle\sum_{i<j} x_i x_j \\ &\cdots \\ \sigma_n(x) &= x_1 \ldots x_n, \end{aligned}$$

and it is well known that these generate the subring of all symmetric polynomials. □

Example 1.6. Suppose G is the alternating group consisting of all even permutation matrices (matrices as in the previous example, that is, with determinant +1). In this case a G-invariant polynomial can be uniquely expressed as the sum of a symmetric and an alternating polynomial:

$$\left\{\begin{matrix}\text{invariant}\\ \text{polynomials}\end{matrix}\right\} \cong \left\{\begin{matrix}\text{symmetric}\\ \text{polynomials}\end{matrix}\right\} \oplus \left\{\begin{matrix}\text{alternating}\\ \text{polynomials}\end{matrix}\right\}.$$