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B. BOLLOBAS, F. KIRWAN, P. SARNAK, C.T.C. WALL

135 Solitons

T. Miwa
Research Institute for Mathematical Sciences
Kyoto University

M. Jimbo
Kyoto University

E. Date
University of Osaka

Translated by Miles Reid

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Contents

Preface		*page* vii
1	The KdV equation and its symmetries	1
1.1	Symmetries and transformation groups	1
1.2	Symmetries of the KdV equation	5
1.3	The Lax form of an evolution equation (the approach via linear differential equations)	8
Exercises to Chapter 1		9
2	The KdV hierarchy	11
2.1	Pseudodifferential operators	11
2.2	Higher order KdV equations	13
2.3	Infinitely many commuting symmetries	14
2.4	The KP hierarchy	16
Exercises to Chapter 2		18
3	The Hirota equation and vertex operators	19
3.1	The Hirota derivative	19
3.2	n-Solitons	22
3.3	Vertex operators	24
3.4	The bilinear identity	28
Exercises to Chapter 3		31
4	The calculus of Fermions	32
4.1	The Bosonic algebra of differentiation and multiplication	32
4.2	Fermions	34
4.3	The Fock representation	35
4.4	Duality, charge and energy	37
4.5	Wick's theorem	40
Exercises to Chapter 4		41
5	The Boson–Fermion correspondence	43
5.1	Using generating functions	43

5.2	The normal product	44
5.3	Realising the Bosons	46
5.4	Isomorphism of Fock spaces	47
5.5	Realising the Fermions	49
Exercises to Chapter 5		52
6	Transformation groups and tau functions	53
6.1	Group actions and orbits	53
6.2	The Lie algebra $\mathfrak{gl}(\infty)$ of quadratic expressions	54
6.3	The transformation group of the KP hierarchy	58
Exercises to Chapter 6		60
7	The transformation group of the KdV equation	61
7.1	KP hierarchy versus KdV hierarchy	61
7.2	The Transformation group of the KdV equation	63
Exercises to Chapter 7		65
8	Finite dimensional Grassmannians and Plücker relations	66
8.1	Finite dimensional Grassmannians	66
8.2	Plücker coordinates	69
8.3	Plücker relations	71
Exercises to Chapter 8		75
9	Infinite dimensional Grassmannians	76
9.1	The case of finite dimensional Fock space	76
9.2	Description of the vacuum orbit	80
9.3	Young diagrams and character polynomials	82
Exercises to Chapter 9		87
10	The bilinear identity revisited	88
10.1	The bilinear identity and the Plücker relations	88
10.2	Plücker relations and the Hirota equation	90
Exercises to Chapter 10		93
Solutions to exercises		94
Bibliography		103
Index		107

Preface

Waves and vibrations are among the most basic forms of motion, and their study goes back a very long way. Small amplitude waves are described mathematically by a linear differential equation, and their behaviour can be studied in detail. In contrast, when the amplitude is not restricted to being small, the differential equation becomes nonlinear, and its analysis becomes in general an extremely difficult problem.

An example of a nonlinear wave equation is the model

$$\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 \tag{1}$$

for a shallow water wave. This equation was proposed by the physicists Korteweg and de Vries at the end of the nineteenth century, and is now called the KdV equation. In particular, if we now assume a solution in the form of a travelling wave $u(x,t) = f(x-ct)$ then (1) can be integrated: imposing the boundary conditions at large distances that $u(x,t)$ tends to 0 sufficiently fast as $x \to \pm\infty$, we find the exact solution

$$u_1(x,t) = \frac{c}{2}sech^2\left(\frac{\sqrt{c}}{2}(x-ct+\delta)\right), \tag{2}$$

where δ is a constant of integration. The motion described by this is an isolated wave, localised in a small part of space. In fact, in addition to this solution, (1) is known to have an infinite series of exact solutions $u_2(x,t), u_3(x,t), \ldots$.

These solutions $u_n(x,t)$ contain $2n$ arbitrary parameters c_i, δ_i, and in the distant past $t \ll 0$ and the distant future $t \gg 0$ they behave just like a superposition of independent isolated waves of the form (2). The isolated waves can overtake or collide with one another in finite time, but they revert after the collision to their individual independent state (except for a possible phase change), and are transmitted without annihilation. A wave motion with this special type of particle-like behaviour is called a *soliton*, and a solution $u_n(x,t)$ representing n isolated waves is called an *n-soliton.*

For a linear differential equation, the *principle of superposition* says that if special solutions u_i are known for $i = 1, \ldots, N$, then other solutions containing arbitrary constants can be made as linear combinations $\sum_{i=1}^{N} c_i u_i$. The principle of superposition does not apply to the KdV equation, because it is nonlinear. The fact that, despite this, there exist exact solutions containing an arbitrary number of parameters, is a remarkable and exceptional phenomenon, and it suggests that the KdV equation occupies a special position among general nonlinear differential equations.

In classical mechanics, there is a notion of a *completely integrable system* (we say simply *integrable system* for short): consider a mechanical system with f degrees of freedom

$$\frac{\mathrm{d}q_i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i}, \quad \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial H}{\partial q_i} \qquad \text{for } i = 1, \ldots, f. \tag{3}$$

Here H is a Hamiltonian. We say that (3) is a completely integrable system if it has f independent first integrals $F_1(q,p) = H(p,q), \ldots, F_f(q,p)$. When this holds, the general solution of (3) can be obtained by solving $F_i(q,p) = C_i$ for $i = 1, \ldots, f$, where C_i are arbitrary constants. Now it is known the KdV equation can in fact be interpreted as an integrable system in this sense, but having infinitely many degrees of freedom. The existence of infinitely many exact solutions such as the soliton solutions is a reflection of this complete integrability.

Although these remarkable properties of the KdV equation were considered as an isolated special phenomenon when they were first discovered, their universal nature became gradually more apparent in rapidly developing research from the late 1960s onwards. At present, a huge number of concrete examples of integrable nonlinear differential (and difference) equations are known. These are also quite generally called soliton equations. A model example of these is the Toda lattice discovered by Morikazu TODA. Many techniques for finding exact solutions of these equations have also been discovered: inverse scattering theory which solves the initial value problem, the bilinear method initiated by Ryogo HIROTA, the theory of quasiperiodic solutions based on Riemann surfaces and theta functions, etc.

At the same time, classical results that had remained long buried came to be viewed in a new light: the applications of theta functions to classical mechanics, nonlinear differential equations arising in the differential geometry of surfaces, the study of commutative subrings of rings of differential operators, and so on. One could say that the KP (Kadomstev–Petviashvili) equations, which generalise the KdV equations, the Toda equation, the Hirota derivative and so on, had already existed in a different form. The theory of integrable systems was confirmed as a paradigm providing a unified viewpoint on these various results.

What is the guiding principle behind the complete integrability of all these systems? In a word, it is the extremely high degree of *symmetry* hidden in the system. For 'high degree of symmetry', we could equally well say 'action of a huge transformation group'. The aim of this book is to use the KdV and KP equations as material to introduce the idea of an infinite dimensional transformation group acting on spaces of solutions of integrable systems. Mikio SATO discovered that the totality of solutions of the KP equations form an infinite dimensional Grassmannian, and established the algebraic structure theory of completely integrable systems. Our aim is to explain the essence of this theory of Sato, together with development of these ideas in the research of Masaki KASHIWARA and the present authors, without going into all the details. We leave to the reader's kind judgment the extent to which we have succeeded in our aim.

As far as prerequisites are concerned, we have tried to write the book so that it can be read by a student with a knowledge of differential and integral calculus, linear algebra and elementary complex analysis (up to the calculus of residues).

Finally, we would like to thank Shigeki SUGIMOTO, Takeshi SUZUKI and Masato HAYASHI for reading the manuscript and making useful comments.

Tetsuji MIWA
Michio JIMBO
Etsuro DATE

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