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The KdV equation and its symmetries

We look for symmetries of the KdV equation taking the form of infinitesimal transformations by a nonlinear evolution equation. The KdV equation is itself a nonlinear evolution equation, but we will see how to derive it in terms of compatibility conditions between *linear* equations.

The best possible compass to guide us in mathematics and the natural sciences is the notion of *symmetry*. Following this compass, up anchor and away over the wide ocean of solitons!

1.1 Symmetries and transformation groups

So then, what is symmetry? For example, consider the symmetries of the circle. One sees fairly intuitively that the circle is taken into itself by either

(1) a rotation around the centre, or
(2) a reflection in a diameter.

How do we express this intuition in precise mathematical terms? In the (x, y) coordinate plane, the circle is given as the set of points satisfying

$$x^2 + y^2 = r^2. \tag{1.1}$$

A rotation of the plane is the linear transformation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{1.2}$$

and a reflection

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{1.3}$$

The linear transformation given by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{1.4}$$

represents a symmetry of the circle if it preserves the expression (1.1). In other words, we say that (1.4) is a *symmetry* of (1.1) if (x', y') is a solution of (1.1) whenever (x, y) is.

Write $T(\theta)$ for the transformation (1.2), and $S(\theta)$ for (1.3). The set of all invertible linear transformations forms a group under composition. In other words, if we define the product of two matrices $T_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ and $T_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$ with nonzero determinant to be the matrix $T_1 \cdot T_2 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$ then the group axioms† are satisfied:

(1) Associativity: $(T_1 \cdot T_2) \cdot T_3 = T_1 \cdot (T_2 \cdot T_3)$.

(2) Existence of the unit: $T \cdot \mathrm{id} = \mathrm{id} \cdot T$; here $\mathrm{id} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

(3) Existence of the inverse: $T \cdot T^{-1} = T^{-1} \circ T = \mathrm{id}$.

Restricting attention in particular to the elements that leave the circle (1.1) invariant, these also form a group. This is called a *transformation group* of the circle. We have $T(\theta) = T(\theta')$, or $S(\theta) = S(\theta')$, if and only if $\theta = \theta' + 2n\pi$, where n is an integer. The group law is given by

$$\left.\begin{aligned} T(\theta_1) \cdot T(\theta_2) &= T(\theta_1 + \theta_2), \\ T(\theta_1) \cdot S(\theta_2) &= S(\theta_2) \cdot T(-\theta_1) = S(\theta_1 + \theta_2), \\ S(\theta_1) \cdot S(\theta_2) &= T(\theta_1 - \theta_2), \end{aligned}\right\} \tag{1.5}$$

By passing from the transformations themselves to the composition rules (1.5), the symmetries of the circle are isolated from their concrete nature as transformations of the plane, and purified into an abstract group law.

Among symmetries of the circle, consider only the rotations $T(\theta)$. When the parameter θ is 0 we have $T(0) = \mathrm{id}$, so that we can view the transformation

$$\begin{pmatrix} x(\theta) \\ y(\theta) \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{1.6}$$

varying together with θ, as the process which takes a given solution (x, y) of the algebraic equation (1.1) continuously around the circle.

† See any textbook on group theory, for example W. Ledermann, *Introduction to group theory*, Oliver and Boyd, 1973, or P.M. Cohn, *Algebra*, Vol. I, Wiley, 1974, Section 3.2.

Differentiating this with respect to θ gives

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\begin{pmatrix} x(\theta) \\ y(\theta) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x(\theta) \\ y(\theta) \end{pmatrix}. \tag{1.7}$$

The transformation (1.6) is completely determined by these equations, together with the initial condition

$$\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}. \tag{1.8}$$

The operator $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is an *infinitesimal generator* of the rotation, in a sense we explain presently. We have the relation

$$T(\theta) = \mathrm{e}^{\theta\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}.$$

Expanding this with θ as a small parameter gives

$$T(\theta) = 1 + \theta\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + O(\theta^2). \tag{1.9}$$

More generally, if $R(\theta)$ is a transformation depending on one parameter θ and satisfying $R(\theta_1 + \theta_2) = R(\theta_1)R(\theta_2)$, and we have $R(\theta) = 1 + \theta X + O(\theta^2)$, then we say that X is the *infinitesimal generator* of the one parameter transformation $R(\theta)$, and we set $R(\theta) = \mathrm{e}^{\theta X}$. (Compare the discussion of Lie algebras at the end of this section.) If we think of $R(\theta)$ as acting on an initial object f, and we write $f^R(\theta) = R(\theta)f$, then

$$\frac{\mathrm{d}f^R}{\mathrm{d}\theta} = \frac{\mathrm{d}}{\mathrm{d}\theta}R(\theta)f = Xf^R(\theta). \tag{1.10}$$

We are primarily interested in transformations and infinitesimal transformations acting on functions. For example, for a function of two variables $f(x, y)$, consider the differential equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - r^2\right) f(x, y) = 0, \tag{1.11}$$

where r is a constant, independent of (x, y). Whereas for the algebraic equation (1.1) we looked for solutions in the 2 dimensional (x, y) plane, for the differential equation (1.11), the solution $f(x, y)$ should live in the infinite dimensional vector space of functions of two variables. Now a rotation of the (x, y) plane induces an action $g \mapsto T(\theta)g$ on the space of functions g of two variables (x, y) by the formula

$$\big(T(\theta)g\big)(x, y) = g\big(x(-\theta), y(-\theta)\big)$$

(compare (1.6)). Or by considering $f^T(x, y; \theta) = \big(T(\theta)f\big)(x, y)$, we can write this as an infinitesimal transformation, giving

$$\frac{\partial}{\partial \theta} f^T(x, y; \theta) = \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right) f^T(x, y; \theta).$$

Thus the operator $x(\partial/\partial y) - y(\partial/\partial x)$ is an infinitesimal generator of the transformation. The equation (1.11) has rotational symmetry; thus if f is a solution of (1.11), so is $T(\theta)f$. The equation (1.11) is also invariant under parallel translation $(x, y) \mapsto (x + a, y + b)$. Expressing parallel translation also as an infinitesimal transformation gives

$$f(x + a, y + b) = \mathrm{e}^{a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}} f(x, y), \tag{1.12}$$

which is just the Taylor expansion of f around (x, y). (We note here that this equation is often used in what follows. Compare also Exercise 1.1.)

For use in future chapters, we now want to give a brief treatment of Lie algebras. Suppose that X and Y are linear differential operators, and are the infinitesimal generators of operators $\mathrm{e}^{\theta X}$, $\mathrm{e}^{\theta Y}$; we consider the product of $\mathrm{e}^{\theta X}$ and $\mathrm{e}^{\theta Y}$. In what follows we use the notation

$$[A, B] = AB - BA$$

for the *commutator bracket* of the operators A and B. A calculation shows that

$$\mathrm{e}^{\theta X}\mathrm{e}^{\theta Y} = \mathrm{e}^{\theta X + \theta Y + (1/2)\theta^2[X,Y] + (1/12)\theta^3[X-Y,[X,Y]] + \cdots}.$$

Here, in the exponent on the right hand side, the $+\cdots$ means terms of higher order in θ; it can be shown that each of these can be written using only commutator brackets $[-,-]$, without any products. If $[X, Y] = 0$, that is, if X and Y commute, then $\mathrm{e}^{\theta X}\mathrm{e}^{\theta Y} = \mathrm{e}^{\theta(X+Y)}$, so that the composite $\mathrm{e}^{\theta X}\mathrm{e}^{\theta Y}$ of the two transformations coincides with the transformations $\mathrm{e}^{\theta(X+Y)}$ corresponding to $X + Y$. In general these two do not coincide, but the difference between them can be computed by knowing the commutator bracket of the infinitesimal generators.

A *Lie algebra* is a vector space $\mathfrak{g}$, together with a law which associates to any two elements $X, Y \in \mathfrak{g}$ a bracket $[X, Y] \in \mathfrak{g}$, satisfying

$$\begin{aligned} &(1) \qquad [X, Y] = -[Y, X], \\ &(2) \qquad [[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0, \\ &(3) \qquad [\alpha X + \beta Y, Z] = \alpha[X, Z] + \beta[Y, Z]. \end{aligned} \tag{1.13}$$

(Here α, β are coefficients acting by scalar multiplication in the vector space $\mathfrak{g}$.) If we ignore worries about convergence, then for a Lie algebra $\mathfrak{g}$, the set of transformations

$$G = \left\{ \mathrm{e}^{X} \;\middle|\; X \in \mathfrak{g} \right\}$$

is a group. To run ahead of ourselves, we note that when treating infinite dimensional symmetries, as we do in soliton theory, it often happens that the Lie algebra $\mathfrak{g}$ is comparatively easy to deal with, even in cases where handling the transformation group G might lead to difficulties.

1.2 Symmetries of the KdV equation

As explained above, there are two different contexts in which rotations are generated by an infinitesimal linear transformation:

$$\text{in 2 dimensional space by } \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix};$$

$$\text{in an infinite dimensional space by } x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}.$$

We can also consider nonlinear infinitesimal transformations. For functions of two variables $u(x,t)$, consider the differential equation

$$\frac{\partial u}{\partial t} = u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3}. \tag{1.14}$$

This is the KdV equation, the main theme of this chapter. Here we take the coefficients of $u(\partial u/\partial x)$ and $\partial^3 u/\partial x^3$ to be equal to 1, but we can make them into arbitrary nonzero constants by multiplying t, x and u by constant scaling factors. All of these are also called KdV equations. This equation describes an infinitesimal transformation in time t acting on a function u of x by the operator

$$K(u) = u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3}. \tag{1.15}$$

Quite generally, an equation of the form $\partial u/\partial t = K(u)$ is called an *evolution equation*. The equation is said to be linear or nonlinear depending on the nature of the operator $K(u)$. If $K(u)$ is linear then $u \mapsto K(u)$ is just the infinitesimal generator described in Section 1.1.

From now on, we say that $K(u)$ is an infinitesimal generator also in the nonlinear case. We interpret the evolution equation, including the nonlinear case, as given by infinitesimal transformation of functions, and search for symmetries of the KdV equation among these. We pose the problem as follows: does the KdV equation

$$\frac{\partial u}{\partial t} = K(u) \tag{1.16}$$

have a symmetry of the form

$$\frac{\partial u}{\partial s} = \widehat{K}(u)? \tag{1.17}$$

What does it mean to say that (1.17) is a symmetry of (1.16)? Consider a function of three variables $u(x,t,s)$. In what follows, for simplicity, we write derivatives (and higher order derivatives) as

$$(\partial u/\partial t) = u_t, \quad \partial^3 u/\partial x^3 = u_{xxx} = u_{3x},$$

and so on. A polynomial in u and its x-derivatives u_x, u_{xx}, u_{3x}, ... is called a *differential polynomial* in u with respect to x. For example, (1.15) is a differential polynomial in u.

Let $\widehat{K}(u)$ be a differential polynomial in u. We consider (1.17) as an evolution equation in time s, and suppose that it can be solved with the given initial value $u(x,t,s=0)$. In other words, starting from the function of two variables $u(x,t,s=0)$ at time $s=0$, and solving (1.17), we get the function $u(x,t,s=\Delta s)$ at time Δs. To say that (1.17) gives a symmetry of the KdV equation means exactly that if $u(x,t,s=0)$ is a solution of (1.16) at time $s=0$, then so is $u(x,t,s)$ at any time s.

Treating t and s on an equal footing reinterprets the question as follows: suppose that t and s are two independent time variables, and that we are given a function $u(x,t=0,s=0)$ when both $t=0$ and $s=0$. Then there are two methods to determine the function $u(x,\Delta s,\Delta t)$ at time $(\Delta t,\Delta s)$, as shown in the following diagram:

$$\begin{tikzcd}
u(x,t=\Delta t,s=0) \arrow[r] & u(x,t=\Delta t,s=\Delta s) \\
u(x,t=0,s=0) \arrow[u] \arrow[r] \arrow[ur, bend left, "A"] \arrow[ur, bend right, "B"'] & u(x,t=0,s=\Delta s) \arrow[u]
\end{tikzcd} \tag{1.18}$$

In this diagram, the up arrows stand for solving (1.16), and the right arrows for solving (1.17). The composite arrow A goes first up, then across; whereas B goes first across, then up. If A and B give the same result then (1.17) is clearly a symmetry of (1.16). Passing to the limiting case when Δt, Δs are very small in (1.18), we see that for A = B to hold, we must have

$$\frac{\partial}{\partial s}K(u) = \frac{\partial}{\partial t}\widehat{K}(u). \tag{1.19}$$

Now does (1.19) hold for an arbitrary choice of $\widehat{K}(u)$? For example, if we try setting $\widehat{K}(u) = u^2$, we get

$$\begin{aligned} \text{left}-\text{handside} &= (uu_x + u_{3x})_s = u^2u_x + u(u^2)_x + (u^2)_{3x} \\ &= 3u^2u_x + 6u_xu_{xx} + 2uu_{3x}, \\ \text{right}-\text{handside} &= (u^2)_t = 2u^2u_x + 2uu_{3x}. \end{aligned}$$

So we see that (1.19) fails without the additional condition $0 = u^2u_x + 6u_xu_{xx}$. Thus for arbitrary choices of $\widehat{K}(u)$, (1.16) and (1.17) are not compatible, so that (1.17) is not a symmetry of (1.16).

If in (1.15) we give u degree 2, u_x degree 3 and u_{xx} degree 4, then the right-hand side is homogeneous of degree 5. Although we do not explain the reason behind it, in searching for symmetries, there is no loss of generality in our argument below in restricting the infinitesimal generating operator to be a homogeneous differential polynomial.

The general form of a homogeneous differential polynomial of degree 7 is

$$C_1u^2u_x + C_2uu_{3x} + C_3u_xu_{xx} + C_4u_{5x}. \tag{1.20}$$

Substituting (1.20) for $\widehat{K}(u)$ in (1.17) and assuming $C_1 = 1$, we see, after a similar but lengthy calculation, that the coefficients C_i are uniquely determined by the condition that (1.16) and (1.17) are compatible. In fact, we get

$$\frac{\partial u}{\partial s} = u^2u_x + 2uu_{3x} + 4u_xu_{xx} + \frac{6}{5}u_{5x}. \tag{1.21}$$

Remark 1.1 *There do exist homogeneous differential polynomials of degrees 3 (respectively 5), but the resulting symmetries of $u(x,t)$ correspond simply to the parallel translation $x \mapsto x + s$ (respectively $t \mapsto t + s$).*

If we set to work to calculate more systematically, we would find that there apparently exists just one symmetry in each odd degree. Now, how can we carry out the argument for every odd number? See Exercise 1.2 for an example other than the KdV equation.

1.3 The Lax form of an evolution equation (the approach via linear differential equations)

Consider the linear differential equation

$$Pw = k^2 w, \quad \text{where} \quad P = \frac{\partial^2}{\partial x^2} + u. \tag{1.22}$$

We think of u as given as a function of x, and P as an operator acting on functions of x. Thus k^2 is an *eigenvalue* of P, and k is called the *spectral* variable. If $u \equiv 0$ then $w = \mathrm{e}^{kx}$ is one solution, but we can also look for solutions in the general case as formal power series of the form

$$w = \mathrm{e}^{kx}\left(w_0 + \frac{w_1}{k} + \frac{w_2}{k^2} + \cdots\right). \tag{1.23}$$

Here *formal* means that we do not necessarily require the power series to converge. Substituting (1.23) in (1.22) gives

$$\frac{\partial w_0}{\partial x} = 0 \quad \text{and} \quad 2\frac{\partial w_j}{\partial x} + \frac{\partial^2 w_{j-1}}{\partial x^2} + u w_{j-1} = 0 \quad \text{for } j \geq 1.$$

Assuming that $w_0 \equiv 1$, the w_j can be determined successively (up to constants of integration) by integrating with respect to x.

We now introduce a time variable t, and allow the given function $u = u(x,t)$ to vary with t. We want to solve (1.22) in terms of a time evolution of w with a linear operator. The operator P in (1.22) is a second order differential operator, so this time we try looking for a third order differential operator

$$\frac{\partial w}{\partial t} = Bw, \quad \text{where} \quad B = \frac{\partial^3}{\partial x^3} + b_1\frac{\partial}{\partial x} + b_2. \tag{1.24}$$

Solving this gives a function $w(x,t;k)$ of two variables x, t for any fixed value of k. We know that (for u independent of k) at time $t = 0$, the function $w(x, t = 0; k)$ satisfies (1.22). Does this continue to hold at other times t? (For this to make sense, $u = u(x,t)$ must also be independent of k.) If (1.22) holds then differentiating both sides with respect to t gives

$$\left(\frac{\partial P}{\partial t} + [P,B]\right) w = 0. \tag{1.25}$$

Here $[P,B] = PB - BP$ is the commutator bracket of the differential operators P and B, and $\partial P/\partial t = \partial u/\partial t$, where u is the given function. Thus (1.25) only involves derivatives with respect to x, so is an ordinary differential equation (independent of k). If (1.25) holds for an arbitrary value of the eigenvalue k then the ODE (1.25) has infinitely many independent solutions. This is impossible unless the differential equation is trivial. Thus we must have

$$\frac{\partial P}{\partial t} + [P, B] = 0. \tag{1.26}$$

Writing this out as conditions on the coefficients u of P and b_1, b_2 of B gives

$$\begin{aligned} b_1 &= \frac{3}{2}u, \\ b_2 &= \frac{3}{4}u_x, \\ \frac{\partial u}{\partial t} &= \frac{3}{2}uu_x + \frac{1}{4}u_{3x}. \end{aligned} \tag{1.27}$$

Here we are solving under the condition that u, b_1, b_2 and their x-derivatives tend to 0 as $x \to \pm\infty$. Thus (1.22) and (1.24) are compatible only if $u(x,t)$ is a solution of the KdV equation. The compatibility condition (1.26) is equivalent to the KdV equation, and is called the *Lax form* of the KdV equation.

We summarise the above argument schematically:

$$\boxed{\begin{array}{l} \text{linear system of equations:} \quad Pw = k^2 w \quad \text{and} \quad \dfrac{\partial w}{\partial t} = Bw \\[2ex] \qquad\qquad\qquad \Big\Downarrow \begin{array}{l}\text{compatibility}\\ \text{condition}\end{array} \\[2ex] \text{Lax representation of the KdV equation:} \quad \dfrac{\partial P}{\partial t} = [B, P]. \end{array}} \tag{1.28}$$

Replacing the third order linear differential operator B in x with linear differential operators of higher order gives rise to nonlinear evolution equations called the *higher order KdV equations*, which involve higher order derivatives. To see this clearly, we carry out a few algebraic preliminaries in the next chapter. See Exercise 1.3.

Exercises to Chapter 1

1.1. What is the function generated from $f(x) = x$ by the infinitesimal transformation $x^2\partial/\partial x$?

1.2. Determine a symmetry of the equation

$$\frac{\partial u}{\partial t} = u^2 u_x + u_{xxx}.$$

[Hint: Set $\widehat{K}(u) = Au^4u_x + Bu^2u_{3x} + Cuu_xu_{2x} + Du_x^3 + Eu_{5x}$, calculate $(\partial/\partial t)(\widehat{K}(u))$ and $(\partial/\partial s)(K(u))$ by the method indicated after (1.19), then equate coefficients to determine A, B, C, D, E.]

1.3. What equation do you get from the Lax equation (1.26) if you swap the roles of P (1.22) and B (1.24)?