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# TOEPLITZ MATRICES AND OPERATORS

The theory of Toeplitz matrices and operators is a vital part of modern analysis, with applications to moment problems, orthogonal polynomials, approximation theory, integral equations, bounded- and vanishing-mean oscillations, and asymptotic methods for large structured determinants, among others.

This friendly introduction to Toeplitz theory covers the classical spectral theory of Toeplitz forms and Wiener–Hopf integral operators and their manifestations throughout modern functional analysis. Numerous solved exercises illustrate the results of the main text and introduce subsidiary topics, including recent developments. Each chapter ends with a survey of the present state of the theory, making this a valuable work for the beginning graduate student and established researcher alike. With biographies of the principal creators of the theory and historical context also woven into the text, this book is a complete source on Toeplitz theory.

**Nikolaï Nikolski** is Professor Emeritus at the Université de Bordeaux, working primarily in analysis and operator theory. He has been co-editor of four international journals, editor of more than 15 books, and has published numerous articles and research monographs. He has also supervised 26 Ph.D. students, including three Salem Prize winners. Professor Nikolski was elected Fellow of the American Mathematical Society (AMS) in 2013 and received the Prix Ampère of the French Academy of Sciences in 2010.

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# Toeplitz Matrices and Operators

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CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE
UNIVERSITY PRESS

Shaftesbury Road, Cambridge CB2 8EA, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

314–321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre, New Delhi – 110025, India

103 Penang Road, #05–06/07, Visioncrest Commercial, Singapore 238467

Cambridge University Press is part of Cambridge University Press & Assessment,
a department of the University of Cambridge.

We share the University's mission to contribute to society through the pursuit of
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www.cambridge.org
Information on this title: www.cambridge.org/9781107198500

DOI: 10.1017/9781108182577



First published 2020

*A catalogue record for this publication is available from the British Library*

ISBN 978-1-107-19850-0 Hardback

Уничтожайте рукописи,
но сохраняйте то, что вы начертали сбоку,
от скуки, от неуменья, и как бы во сне.
Эти второстепенные и мимовольные создания вашей фантазии не
пропадут в мире

Destroy your manuscript,
but save whatever you have inscribed in the margin
out of boredom, out of helplessness, and, as it were, in a dream.
These secondary and involuntary creations of your fantasy will not be lost in the world.

*The Egyptian Stamp* (1928)
Osip Mandelstam

# Contents

# Preface

*Par ma fois! . . . je dis de la prose*
*sans que j'en susse rien, et je vous suis*
*le plus obligé du monde de m'avoir appris cela.*

Good heavens! . . . I have been speaking prose
without knowing anything about it, and I am
much obliged to you for having taught me that.

Monsieur Jourdain (1670)

As in Molière's *Le Bourgeois gentilhomme* with his prose, we often speak the language of Toeplitz and Hankel matrices/operators without realising it; but it would be better if we did it knowingly and in a technically correct manner.

The introduction to Toeplitz operators and matrices proposed in this text concerns the matrix and integral transforms defined by a "kernel" (a matrix or a function of two variables) with constant diagonals. In particular, a sequence of complex numbers $(c_k)_{k\in\mathbb{Z}}$ defines a *Toeplitz matrix*

$$T = \begin{pmatrix} c_0 & c_{-1} & c_{-2} & c_{-3} & \dots & \dots \\ c_1 & c_0 & c_{-1} & c_{-2} & \dots & \dots \\ c_2 & c_1 & c_0 & c_{-1} & \dots & \dots \\ c_3 & c_2 & c_1 & c_0 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

and a *Hankel matrix*

$$\Gamma = \begin{pmatrix} c_{-1} & c_{-2} & c_{-3} & c_{-4} & \dots & \dots \\ c_{-2} & c_{-3} & c_{-4} & c_{-5} & \dots & \dots \\ c_{-3} & c_{-4} & c_{-5} & c_{-6} & \dots & \dots \\ c_{-4} & c_{-5} & c_{-6} & c_{-7} & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

An operator (mapping) that has a Toeplitz (respectively, Hankel) matrix with respect to a basis is called a *Toeplitz* (respectively, *Hankel*) operator. For deep reasons (deep in two senses: hidden deep beneath the surface of mathematical facts, and deep in terms of their power), these transforms have played an exceptional role in contemporary mathematics. The set of problems and the basis of the techniques associated with these transforms were formulated by such giants of mathematics as Bernhard Riemann, David Hilbert, Norbert Wiener, George Birkhoff, and Otto Toeplitz.

This book is an introduction to a highly dynamic domain of modern analysis based on the techniques of Hardy spaces. To make it independent we provide a thorough overview of these spaces in Appendix F, and each time we reiterate the principal facts of Hardy theory immediately before their use. A complete presentation of Hardy spaces, the closest in style to the present work, can be found in *Espaces de Hardy* (Éditions Belin, 2012) by the same author, also translated as *Hardy Spaces* (Cambridge University Press, 2019). The current text, like the first, corresponds to a course at Master's level, given several times at the Université de Bordeaux during the years 1991–2011. Numerous resolved exercises show how the techniques developed can be put into action and extend the scope of the theory.

The somewhat hidden aspect of this text is the following. It is devoted to the study of integral and matrix operators with kernels (or matrices) depending on the difference of arguments, which seems, at first glance, to be a rather specialized subject. But a second glance leads to the discovery that many classical results of analysis and its applications rely directly on the operators known as "Toeplitz" and their "siblings," the "Hankel" operators, which are so intimately associated with those of Toeplitz that the pair are often referred to as the "Ha-plitz operators/matrices." This area of analysis includes Wiener's filtering problems, the statistical physics of gases, diverse moment problems, ergodic properties of random processes, complex interpolation, etc. The goal of this text is to present the diversity of Toeplitz/Hankel techniques and to draw conclusions from the "inexplicable efficacy" of Toeplitz (and Hankel) operators.

The *prerequisites* are standard courses on functional analysis (or Hilbert/Banach spaces) along with a few elements of complex analysis and a certain familiarity with Hardy spaces. A summary/overview of all these basic notions (as well as the notation used) can be found in the Appendices at the end of the book.

More precisely, Appendices A–D provide the definitions and notation of basic analysis (at undergraduate level), whereas Appendix E provides a short but complete presentation (including proofs) of a less well-known theory of functional analysis – that of “Fredholm operators.” Appendix F is a summary of the theory of Hardy spaces; the text by the same author on this subject, *Espaces de Hardy* (*Hardy Spaces*), is cited here as [Nikolski, 2019].

Within the text, there are also numerous historical references – on the subjects developed, their creators, and their diverse situations. We only hope that these “asides” will help the reader to better appreciate the mathematical methods presented and their efficacy, as well as the dramaturgy of mathematics (and mathematical life).

Each chapter contains exercises and their solutions (155 in total) at different levels. To use a musical metaphor of Israel Glazman and Yuri Lyubich [Glazman and Lyubich, 1969], they range from exercises on open strings up to virtuoso pieces using double harmonics (“double flageolet tones”). In particular, the series of exercises in each chapter (with the exception of Chapter 1) begin with Basic Exercises accessible at Master’s level or for preparing students for the French *agrégations* exams (a competitive exam to attain the highest teaching diploma).

Each chapter concludes with a section entitled “Notes and Remarks,” which discusses the history of the subjects treated, certain recent results, and (on occasion) some open questions; this discussion is destined primarily for the more experienced reader. For an appreciation of this type of text in a poetic form, see the maxim on page v, due to Osip Mandelstam, the greatest Russian poet of the twentieth century.

*Chapter 1* plays the role of a detailed but informal introduction: a description of sources of inspiration for the (future) theory, the principal components of its current state, as well as a panorama of its applications and the history of its evolution during the twentieth century.

*Chapter 2* establishes the basic contours of the theory of Hankel/Toeplitz operators on the circle $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$.

*Chapter 3* is devoted to the spectral theory of Toeplitz operators.

*Chapter 4* explains the unitary equivalence of the theory on $\mathbb{T}$ with the classical theory of Wiener–Hopf equations on $\mathbb{R}$, as well as the Riemann–Hilbert problem.

*Chapter 5* treats some properties of finite Toeplitz matrices (inversion, links with trigonometric moments, the approximation of infinite Toeplitz matrices, the asymptotic distribution of spectra).

The reader will soon become aware that this handbook constitutes a rather elementary introduction to Toeplitz matrices and operators based on the theory of Hardy spaces. It can thus be regarded as a source of basic knowledge (a "First elements of ..."). Nevertheless, in principle, a student reaching the end of this book will be capable of embarking upon a project of independent research (the author can affirm this by positive experiences with numerous students). For such an endeavor, the aid of experts will be needed: this can be found in the dozens of existing monographs devoted to Toeplitz operators and matrices, to Hardy spaces, and to the "hard analysis" that was developed around them. Some of this torrent of literature is mentioned in the "Notes and Remarks" sections.

Good luck!

# Acknowledgments

## Acknowledgments for the English Edition

The author warmly thanks the translators Danièle and Greg Gibbons for their high-quality work, for thorough attention to all shades of meaning of the French text, and for friendly collaboration at all stages of the work.

The author is also sincerely grateful to Cambridge University Press for including the book in this prestigious series, and to the entire editorial team for highly professional preparation of the manuscript and for patience during his numerous hold-ups due to varying circumstances.

Élancourt,

February 4, 2019

## Acknowledgments for the French Edition

The work on the final draft of this text was in part supported by the research project "Spaces of Analytic Functions and Singular Integrals" of the University of St. Petersburg, RSF Grant 14-41-00010.

I am truly grateful to Éric Charpentier, my friend and colleague at the Université de Bordeaux, who – as with my preceding book (*Espaces de Hardy*, Belin, 2012) – sacrificed much of his time for a thorough and effective review of the complete text. Without his invaluable and generous aid the text would never have seen the light of day.

I warmly thank Albrecht Böttcher, the great expert in the Toeplitz domain, who took the time to read the manuscript and whose profound and nourishing comments aided me in polishing the text. I am also beholden to several of my colleagues for *ad hoc* consultations on the subjects of the book, and especially to Anton Baranov of St. Petersburg for his remarks and suggestions concerning the editing.

I also address my cordial thanks to Gilles Godefroy (Institut de Mathématiques de Jussieu, Université Paris VI) who, at a delicate moment of the project, rescued it from a dead end brought on by circumstances beyond my control, by recommending the publisher Calvage et Mounet. I would of course also like to warmly thank the publisher (in particular Alain Debreil and Rachid Mneimné) for the painstaking and highly qualified attention they paid to the present manuscript, as well as for the friendly atmosphere that marked our collaboration on this occasion.

And finally, but foremost, I think of my large family – Pascale, Laure, Ivan, Jeanne, and Alekseï – who stoically supported my prolonged isolation while I struggled with my text, and who encouraged me in my moments of doubt.

Élancourt,
November 2016

# Biographies

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