

Cambridge University Press & Assessment
978-1-107-19850-0 — Toeplitz Matrices and Operators
Nikolaï Nikolski, Translated by Danièle Gibbons, Greg Gibbons
Frontmatter
[More Information](#)

CAMBRIDGE STUDIES IN ADVANCED MATHEMATICS 182

Editorial Board

B. BOLLOBÁS, W. FULTON, F. KIRWAN,
P. SARNAK, B. SIMON, B. TOTARO

TOEPLITZ MATRICES AND OPERATORS

The theory of Toeplitz matrices and operators is a vital part of modern analysis, with applications to moment problems, orthogonal polynomials, approximation theory, integral equations, bounded- and vanishing-mean oscillations, and asymptotic methods for large structured determinants, among others.

This friendly introduction to Toeplitz theory covers the classical spectral theory of Toeplitz forms and Wiener–Hopf integral operators and their manifestations throughout modern functional analysis. Numerous solved exercises illustrate the results of the main text and introduce subsidiary topics, including recent developments. Each chapter ends with a survey of the present state of the theory, making this a valuable work for the beginning graduate student and established researcher alike. With biographies of the principal creators of the theory and historical context also woven into the text, this book is a complete source on Toeplitz theory.

Nikolaï Nikolski is Professor Emeritus at the Université de Bordeaux, working primarily in analysis and operator theory. He has been co-editor of four international journals, editor of more than 15 books, and has published numerous articles and research monographs. He has also supervised 26 Ph.D. students, including three Salem Prize winners. Professor Nikolski was elected Fellow of the American Mathematical Society (AMS) in 2013 and received the Prix Ampère of the French Academy of Sciences in 2010.

Cambridge University Press & Assessment
 978-1-107-19850-0 — Toeplitz Matrices and Operators
 Nikolaï Nikolski, Translated by Danièle Gibbons, Greg Gibbons
 Frontmatter
[More Information](#)

CAMBRIDGE STUDIES IN ADVANCED MATHEMATICS

Editorial Board

B. Bollobás, W. Fulton, F. Kirwan, P. Sarnak, B. Simon, B. Totaro

All the titles listed below can be obtained from good booksellers or from Cambridge University Press.
 For a complete series listing, visit www.cambridge.org/mathematics.

Already Published

- 145 M. Viana *Lectures on Lyapunov Exponents*
- 146 J.-H. Evertse & K. Györy *Unit Equations in Diophantine Number Theory*
- 147 A. Prasad *Representation Theory*
- 148 S. R. Garcia, J. Mashreghi & W. T. Ross *Introduction to Model Spaces and Their Operators*
- 149 C. Godsil & K. Meagher *Erdős–Ko–Rado Theorems: Algebraic Approaches*
- 150 P. Mattila *Fourier Analysis and Hausdorff Dimension*
- 151 M. Viana & K. Oliveira *Foundations of Ergodic Theory*
- 152 V. I. Paulsen & M. Raghupathi *An Introduction to the Theory of Reproducing Kernel Hilbert Spaces*
- 153 R. Beals & R. Wong *Special Functions and Orthogonal Polynomials*
- 154 V. Jurdjevic *Optimal Control and Geometry: Integrable Systems*
- 155 G. Pisier *Martingales in Banach Spaces*
- 156 C. T. C. Wall *Differential Topology*
- 157 J. C. Robinson, J. L. Rodrigo & W. Sadowski *The Three-Dimensional Navier–Stokes Equations*
- 158 D. Huybrechts *Lectures on K3 Surfaces*
- 159 H. Matsumoto & S. Taniguchi *Stochastic Analysis*
- 160 A. Borodin & G. Olshanski *Representations of the Infinite Symmetric Group*
- 161 P. Webb *Finite Group Representations for the Pure Mathematician*
- 162 C. J. Bishop & Y. Peres *Fractals in Probability and Analysis*
- 163 A. Bovier *Gaussian Processes on Trees*
- 164 P. Schneider *Galois Representations and (φ, Γ) -Modules*
- 165 P. Gille & T. Szamuely *Central Simple Algebras and Galois Cohomology (2nd Edition)*
- 166 D. Li & H. Queffelec *Introduction to Banach Spaces, I*
- 167 D. Li & H. Queffelec *Introduction to Banach Spaces, II*
- 168 J. Carlson, S. Müller-Stach & C. Peters *Period Mappings and Period Domains (2nd Edition)*
- 169 J. M. Landsberg *Geometry and Complexity Theory*
- 170 J. S. Milne *Algebraic Groups*
- 171 J. Gough & J. Kupsch *Quantum Fields and Processes*
- 172 T. Ceccherini-Silberstein, F. Scarabotti & F. Tolli *Discrete Harmonic Analysis*
- 173 P. Garrett *Modern Analysis of Automorphic Forms by Example, I*
- 174 P. Garrett *Modern Analysis of Automorphic Forms by Example, II*
- 175 G. Navarro *Character Theory and the McKay Conjecture*
- 176 P. Fleig, H. P. A. Gustafsson, A. Kleinschmidt & D. Persson *Eisenstein Series and Automorphic Representations*
- 177 E. Peterson *Formal Geometry and Bordism Operators*
- 178 A. Ogus *Lectures on Logarithmic Algebraic Geometry*
- 179 N. Nikolski *Hardy Spaces*
- 180 D.-C. Cisinski *Higher Categories and Homotopical Algebra*
- 181 A. Agrachev, D. Barilari & U. Boscain *A Comprehensive Introduction to Sub-Riemannian Geometry*
- 182 N. Nikolski *Toeplitz Matrices and Operators*
- 183 A. Yekutieli *Derived Categories*
- 184 C. Demeter *Fourier Restriction, Decoupling and Applications*

Cambridge University Press & Assessment
978-1-107-19850-0 — Toeplitz Matrices and Operators
Nikolai Nikolski, Translated by Danièle Gibbons, Greg Gibbons
Frontmatter
[More Information](#)

Toeplitz Matrices and Operators

NIKOLAI NIKOLSKI
Université de Bordeaux

Translated by
DANIÈLE GIBBONS
GREG GIBBONS



Cambridge University Press & Assessment
978-1-107-19850-0 — Toeplitz Matrices and Operators
Nikolai Nikolski, Translated by Danièle Gibbons, Greg Gibbons
Frontmatter
[More Information](#)



Shaftesbury Road, Cambridge CB2 8EA, United Kingdom
One Liberty Plaza, 20th Floor, New York, NY 10006, USA
477 Williamstown Road, Port Melbourne, VIC 3207, Australia
314–321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre, New Delhi – 110025, India
103 Penang Road, #05–06/07, Visioncrest Commercial, Singapore 238467

Cambridge University Press is part of Cambridge University Press & Assessment,
a department of the University of Cambridge.

We share the University's mission to contribute to society through the pursuit of
education, learning and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9781107198500

DOI: 10.1017/9781108182577

© Nikolai Nikolski 2020

This publication is in copyright. Subject to statutory exception and to the provisions
of relevant collective licensing agreements, no reproduction of any part may take
place without the written permission of Cambridge University Press & Assessment.

First published 2020

A catalogue record for this publication is available from the British Library

ISBN 978-1-107-19850-0 Hardback

Cambridge University Press & Assessment has no responsibility for the persistence
or accuracy of URLs for external or third-party internet websites referred to in this
publication and does not guarantee that any content on such websites is, or will
remain, accurate or appropriate.

Every effort has been made to secure necessary permissions to reproduce
copyright material in this work, though in some cases it has proved impossible
to trace copyright holders. If any omissions are brought to our notice,
we will be happy to include appropriate acknowledgements on reprinting.

Уничтожайте рукописи,
но сохраняйте то, что вы начертали сбоку,
от скуки, от неумения, и как бы во сне.
Эти второстепенные и мимовольные создания вашей фантазии не
пропадут в мире

Destroy your manuscript,
but save whatever you have inscribed in the margin
out of boredom, out of helplessness, and, as it were, in a dream.
These secondary and involuntary creations of your fantasy will not be lost in the world.

The Egyptian Stamp (1928)
Osip Mandelstam

Contents

<i>Preface</i>	<i>page</i> xiii
<i>Acknowledgments for the English Edition</i>	xvii
<i>Acknowledgments for the French Edition</i>	xvii
<i>List of Biographies</i>	xix
<i>List of Figures</i>	xxi
1 Why Toeplitz–Hankel? Motivations and Panorama	1
1.1 Latent Maturation: The RHP and SIOs	1
1.1.1 Nineteenth Century: Riemann and Volterra	1
1.1.2 Twentieth Century: David Hilbert	6
1.1.3 George Birkhoff and Henri Poincaré	12
1.2 The Emergence of the Subject: Otto Toeplitz	16
1.3 The Classical Period	20
1.3.1 Gábor Szegő’s Revolution	20
1.3.2 The Wiener and Hopf Integral Operators	21
1.3.3 A New Challenge Arises: The Lenz–Ising Model	23
1.4 The Golden Age and the Drama of Ideas	23
1.4.1 Solomon Mikhlin and the Symbolic Calculus of the SIO	23
1.4.2 The School of Mark Krein	24
1.4.3 Lars Onsager, and Szegő Again	24
1.4.4 Rosenblum, Devinatz, and the Drama of Coincidence	25
1.5 The Parallel/Complementary World of Hankel, and the Post-modern Epoch of the Ha-plitz Operators	26
1.6 Notes and Remarks	30

2	Hankel and Toeplitz: Sibling Operators on the Space H^2	31
2.1	Three Definitions of Toeplitz Operators: The Symbol	31
2.1.1	The Spaces ℓ^2 , L^2 , and H^2	32
2.1.2	Shift (or Translation) Operators	33
2.1.3	Matrix of an Operator	34
2.1.4	Toeplitz Matrices	35
2.1.5	Toeplitz Operators	35
2.1.6	Comment: Three Equivalent Definitions of Toeplitz Operators	39
2.1.7	Examples	39
2.2	Hankel Operators and Their Symbols	41
2.2.1	Hankel Matrices	41
2.2.2	Hankel Operators	41
2.3	Exercises	50
2.3.0	Basic Exercises: Hilbert and Hardy Spaces, and Their Operators	50
2.3.1	Toeplitz Operators, the Berezin Transform, and the RKT (Reproducing Kernel Thesis)	59
2.3.2	The Natural Projection on \mathcal{T}_{L^∞}	62
2.3.3	Toeplitz Operators on $\ell^p(\mathbb{Z}_+)$ and $H^p(\mathbb{T})$	62
2.3.4	The Space $BMO(\mathbb{T})$, the RKT, and the Garsia Norm	64
2.3.5	Compact Hankel Operators and the Spaces $VMO(\mathbb{T})$ and $QC(\mathbb{T})$	68
2.3.6	Finite Rank Hankel Operators (Kronecker, 1881)	70
2.3.7	Hilbert–Schmidt Hankel Operators	72
2.3.8	The Original Proof of Sarason’s Lemma 2.2.7 (Sarason, 1967)	73
2.3.9	Compactness of the Commutators $[P_+, M_\varphi]$ (Power, 1980)	73
2.3.10	The Natural Projection on $\text{Hank}(\ell^2(\mathbb{Z}_+))$	74
2.3.11	Vector-Valued Toeplitz Operators	75
2.3.12	Some Algebraic Properties of Toeplitz/Hankel Operators	76
2.4	Notes and Remarks	78
3	H^2 Theory of Toeplitz Operators	101
3.1	Fredholm Theory of the Toeplitz Algebra	101
3.1.1	The Role of Homotopy	110

3.2	The Simonenko Local Principle	112
3.2.1	Proof of Theorem 3.2.1 (Sarason, 1973)	115
3.2.2	Examples	116
3.3	The Principal Criterion of Invertibility	117
3.3.1	Wiener–Hopf Factorization	124
3.3.2	Upper Bounds for $\ T_\varphi^{-1}\ $	125
3.3.3	A Comment on Wiener–Hopf Factorization	126
3.3.4	First Consequences of the Principal Criterion	128
3.4	Exercises	129
3.4.0	Basic Exercises: Integral and Multiplication Operators	129
3.4.1	Spectral Inclusions	137
3.4.2	Holomorphic Symbols $\varphi \in H^\infty$	138
3.4.3	Fredholm Theory for the Algebra $\text{alg } \mathcal{T}_{H^\infty + C(\mathbb{T})}$	139
3.4.4	$H^\infty + C(\mathbb{T})$ is the Minimal Algebra Containing H^∞ (Hoffman and Singer, 1960)	141
3.4.5	Fredholm Theory for the Algebra $\text{alg } \mathcal{T}_{PC(\mathbb{T})}$	142
3.4.6	A Simplified Local Principle (Simonenko, 1960)	143
3.4.7	Fred(H^2) and Local Sectoriality	143
3.4.8	Multipliers Preserving Fred(H^2)	145
3.4.9	The Toeplitz Algebra $\text{alg } \mathcal{T}_{L^\infty(\mathbb{T})}$: A Necessary Condition	145
3.4.10	Hankel Operators from the Toeplitz Algebra $\text{alg } \mathcal{T}_{L^\infty(\mathbb{T})}$	145
3.4.11	On the Equation $T_\varphi f = 1$ (Another Proof of Theorem 3.3.8)	146
3.4.12	Is There a Regularizer of T_φ in $\mathcal{T}_{L^\infty(\mathbb{T})}$ and/or in $\text{alg } \mathcal{T}_{L^\infty(\mathbb{T})}$?	146
3.4.13	Fredholm Theory for Almost Periodic Symbols	149
3.4.14	Fredholm Operators T_φ with Matrix-Valued Symbols	151
3.4.15	“Truncated” Toeplitz Operators	153
3.5	Notes and Remarks	155
4	Applications: Riemann–Hilbert, Wiener–Hopf, Singular Integral Operators (SIO)	178
4.1	The Riemann–Hilbert Problem and the SIO	178
4.1.1	The RHP and Toeplitz Operators	179
4.1.2	The Hilbert Transform \mathbb{H} and SIOs	180
4.1.3	Comment: Operators and Singular Integral Equations	186

4.2	Toeplitz on $H^2(\mathbb{C}_+)$ and Wiener–Hopf on $L^2(\mathbb{R}_+)$	188
4.2.1	On the Space $H^2(\mathbb{C}_+)$: The Paley–Wiener Theorem	189
4.2.2	Pseudo-Measures and Wiener–Hopf Operators	191
4.2.3	Transfer of Spectral Theory to Wiener–Hopf Operators	196
4.2.4	Classical Wiener–Hopf Equations and Operators	197
4.2.5	Finite Difference Operators	198
4.2.6	Operators W_μ with Causal Measures μ	198
4.2.7	The Hilbert SIO on $L^2(\mathbb{R}_+)$	199
4.3	The Matrix of W_k in the Laguerre ONB	200
4.4	Wiener–Hopf Operators on a Finite Interval	203
4.4.1	Determination of the Symbol	203
4.4.2	W_k^a of Rank 1	204
4.4.3	Bounding the Norm $\ W_k^a\ $ by the Best Extension	204
4.4.4	Example: An Operator W_k^a Bounded but Without Symbol $k \in \mathcal{PM}(\mathbb{R})$ With Support in $[-a, a]$	205
4.4.5	Example: The Volterra Operator	207
4.5	Exercises	207
4.5.0	Basic Exercises: From the Hilbert Singular Operator to the Riesz Transforms (“Method of Rotation”)	207
4.5.1	Sokhotsky–Plemelj Formulas	211
4.5.2	Systems of Equations and Matrix Wiener–Hopf Operators	214
4.5.3	Hankel Operators on $H^2(\mathbb{C}_+)$ and $L^2(\mathbb{R}_+)$	216
4.5.4	Laguerre Polynomials	218
4.5.5	Compact W_k^a Operators	219
4.6	Notes and Remarks	220
5	Toeplitz Matrices: Moments, Spectra, Asymptotics	230
5.1	Positive Definite Toeplitz Matrices, Moment Problems, and Orthogonal Polynomials	230
5.1.1	Proof of Theorem 5.1.1 (Following Stone, 1932)	235
5.1.2	The Truncated TMP: Extension to a Positive Definite Sequence	235
5.1.3	Truncated Toeplitz Operators	236
5.1.4	The Operator Approach to Orthogonal Polynomials (Akhiezer and Krein, 1938)	237
5.1.5	The Truncated TMP: The Approach of Carathéodory (1911) and Szegő (1954)	241

Contents

xi

5.2	Norm of a Toeplitz Matrix	246
5.2.1	Comments and Special Cases	247
5.2.2	Proof of Theorem 5.2.1	251
5.2.3	Proof of Lemma 5.2.2	253
5.3	Inversion of a Toeplitz Matrix	254
5.3.1	Two Matrix Inversion Theorems	254
5.3.2	Comments	256
5.4	Inversion of Toeplitz Operators by the Finite Section Method	258
5.4.1	The Finite Section Method	258
5.4.2	Theorem (IFSM for Toeplitz Operators)	261
5.4.3	Comment: A Counter-Example of Treil (1987)	265
5.5	Theory of Circulants	265
5.5.1	Cyclic Shift	265
5.5.2	Definition of Circulants	267
5.5.3	Basic Properties	267
5.5.4	Spectrum and Diagonalization of Circulants	269
5.5.5	An Inequality of Wirtinger (1904)	269
5.6	Toeplitz Determinants and Asymptotics of Spectra	270
5.6.1	The First Szegő Asymptotic Formula (1915)	271
5.6.2	Equidistribution of Sequences, after Weyl (1910)	271
5.6.3	Asymptotic Distribution of Spectra	276
5.6.4	Asymptotic Distribution Meets the Circulants	280
5.6.5	The Second Term of the Szegő Asymptotics	284
5.6.6	A Formula for Determinant and Trace	291
5.6.7	Some Formulas for $\text{Trace}[T_\varphi, T_\psi]$ (Following Helton and Howe, and Berger and Shaw, 1973)	291
5.6.8	Conclusion	293
5.7	Exercises	294
5.7.0	Basic Exercises: Volumes, Distances, and Approximations	294
5.7.1	Positive Definite Sequences and Holomorphic Functions	299
5.7.2	Semi-Commutators of Finite Toeplitz Matrices	301
5.7.3	Inversion of Wiener–Hopf Operators by the Finite Section Method	301
5.7.4	When the Second Szegő Asymptotics Stabilize (Szegő, 1952)	302
5.7.5	Cauchy Determinants (1841)	304

xii	<i>Contents</i>	
5.7.6	The Second Term of the Asymptotic Distribution of Spectra (Libkind (1972), Widom (1976))	306
5.7.7	The Helton and Howe Formula of Lemma 5.6.9	306
5.7.8	The Formula of Borodin and Okounkov (2000) (and Geronimo and Case (1979))	307
5.8	Notes and Remarks	308
<i>Appendix A</i>	Key Notions of Banach Spaces	329
<i>Appendix B</i>	Key Notions of Hilbert Spaces	333
<i>Appendix C</i>	An Overview of Banach Algebras	339
<i>Appendix D</i>	Linear Operators	348
<i>Appendix E</i>	Fredholm Operators and the Noether Index	359
<i>Appendix F</i>	A Brief Overview of Hardy Spaces	387
	<i>References</i>	395
	<i>Notation</i>	416
	<i>Index</i>	419

Preface

*Par ma fois! . . . je dis de la prose
 sans que j'en susse rien, et je vous suis
 le plus obligé du monde de m'avoir appris cela.*

Good heavens! . . . I have been speaking prose
 without knowing anything about it, and I am
 much obliged to you for having taught me that.

Monsieur Jourdain (1670)

As in Molière’s *Le Bourgeois gentilhomme* with his prose, we often speak the language of Toeplitz and Hankel matrices/operators without realising it; but it would be better if we did it knowingly and in a technically correct manner.

The introduction to Toeplitz operators and matrices proposed in this text concerns the matrix and integral transforms defined by a “kernel” (a matrix or a function of two variables) with constant diagonals. In particular, a sequence of complex numbers $(c_k)_{k \in \mathbb{Z}}$ defines a *Toeplitz matrix*

$$T = \begin{pmatrix} c_0 & c_{-1} & c_{-2} & c_{-3} & \dots & \dots \\ c_1 & c_0 & c_{-1} & c_{-2} & \dots & \dots \\ c_2 & c_1 & c_0 & c_{-1} & \dots & \dots \\ c_3 & c_2 & c_1 & c_0 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

and a *Hankel matrix*

$$\Gamma = \begin{pmatrix} c_{-1} & c_{-2} & c_{-3} & c_{-4} & \dots & \dots \\ c_{-2} & c_{-3} & c_{-4} & c_{-5} & \dots & \dots \\ c_{-3} & c_{-4} & c_{-5} & c_{-6} & \dots & \dots \\ c_{-4} & c_{-5} & c_{-6} & c_{-7} & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

An operator (mapping) that has a Toeplitz (respectively, Hankel) matrix with respect to a basis is called a *Toeplitz* (respectively, *Hankel*) operator. For deep reasons (deep in two senses: hidden deep beneath the surface of mathematical facts, and deep in terms of their power), these transforms have played an exceptional role in contemporary mathematics. The set of problems and the basis of the techniques associated with these transforms were formulated by such giants of mathematics as Bernhard Riemann, David Hilbert, Norbert Wiener, George Birkhoff, and Otto Toeplitz.

This book is an introduction to a highly dynamic domain of modern analysis based on the techniques of Hardy spaces. To make it independent we provide a thorough overview of these spaces in Appendix F, and each time we reiterate the principal facts of Hardy theory immediately before their use. A complete presentation of Hardy spaces, the closest in style to the present work, can be found in *Espaces de Hardy* (Éditions Belin, 2012) by the same author, also translated as *Hardy Spaces* (Cambridge University Press, 2019). The current text, like the first, corresponds to a course at Master's level, given several times at the Université de Bordeaux during the years 1991–2011. Numerous resolved exercises show how the techniques developed can be put into action and extend the scope of the theory.

The somewhat hidden aspect of this text is the following. It is devoted to the study of integral and matrix operators with kernels (or matrices) depending on the difference of arguments, which seems, at first glance, to be a rather specialized subject. But a second glance leads to the discovery that many classical results of analysis and its applications rely directly on the operators known as “Toeplitz” and their “siblings,” the “Hankel” operators, which are so intimately associated with those of Toeplitz that the pair are often referred to as the “Ha-plitz operators/matrices.” This area of analysis includes Wiener's filtering problems, the statistical physics of gases, diverse moment problems, ergodic properties of random processes, complex interpolation, etc. The goal of this text is to present the diversity of Toeplitz/Hankel techniques and to draw conclusions from the “inexplicable efficacy” of Toeplitz (and Hankel) operators.

The *prerequisites* are standard courses on functional analysis (or Hilbert/Banach spaces) along with a few elements of complex analysis and a certain

familiarity with Hardy spaces. A summary/overview of all these basic notions (as well as the notation used) can be found in the Appendices at the end of the book.

More precisely, Appendices A–D provide the definitions and notation of basic analysis (at undergraduate level), whereas Appendix E provides a short but complete presentation (including proofs) of a less well-known theory of functional analysis – that of “Fredholm operators.” Appendix F is a summary of the theory of Hardy spaces; the text by the same author on this subject, *Espaces de Hardy (Hardy Spaces)*, is cited here as [Nikolski, 2019].

Within the text, there are also numerous historical references – on the subjects developed, their creators, and their diverse situations. We only hope that these “asides” will help the reader to better appreciate the mathematical methods presented and their efficacy, as well as the dramaturgy of mathematics (and mathematical life).

Each chapter contains exercises and their solutions (155 in total) at different levels. To use a musical metaphor of Israel Glazman and Yuri Lyubich [Glazman and Lyubich, 1969], they range from exercises on open strings up to virtuoso pieces using double harmonics (“double flageolet tones”). In particular, the series of exercises in each chapter (with the exception of Chapter 1) begin with Basic Exercises accessible at Master’s level or for preparing students for the French *agrégations* exams (a competitive exam to attain the highest teaching diploma).

Each chapter concludes with a section entitled “Notes and Remarks,” which discusses the history of the subjects treated, certain recent results, and (on occasion) some open questions; this discussion is destined primarily for the more experienced reader. For an appreciation of this type of text in a poetic form, see the maxim on page v, due to Osip Mandelstam, the greatest Russian poet of the twentieth century.

Chapter 1 plays the role of a detailed but informal introduction: a description of sources of inspiration for the (future) theory, the principal components of its current state, as well as a panorama of its applications and the history of its evolution during the twentieth century.

Chapter 2 establishes the basic contours of the theory of Hankel/Toeplitz operators on the circle $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$.

Chapter 3 is devoted to the spectral theory of Toeplitz operators.

Chapter 4 explains the unitary equivalence of the theory on \mathbb{T} with the classical theory of Wiener–Hopf equations on \mathbb{R} , as well as the Riemann–Hilbert problem.

Chapter 5 treats some properties of finite Toeplitz matrices (inversion, links with trigonometric moments, the approximation of infinite Toeplitz matrices, the asymptotic distribution of spectra).

The reader will soon become aware that this handbook constitutes a rather elementary introduction to Toeplitz matrices and operators based on the theory of Hardy spaces. It can thus be regarded as a source of basic knowledge (a “First elements of ...”). Nevertheless, in principle, a student reaching the end of this book will be capable of embarking upon a project of independent research (the author can affirm this by positive experiences with numerous students). For such an endeavor, the aid of experts will be needed: this can be found in the dozens of existing monographs devoted to Toeplitz operators and matrices, to Hardy spaces, and to the “hard analysis” that was developed around them. Some of this torrent of literature is mentioned in the “Notes and Remarks” sections.

Good luck!

Acknowledgments

Acknowledgments for the English Edition

The author warmly thanks the translators Danièle and Greg Gibbons for their high-quality work, for thorough attention to all shades of meaning of the French text, and for friendly collaboration at all stages of the work.

The author is also sincerely grateful to Cambridge University Press for including the book in this prestigious series, and to the entire editorial team for highly professional preparation of the manuscript and for patience during his numerous hold-ups due to varying circumstances.

Élancourt,
February 4, 2019

Acknowledgments for the French Edition

The work on the final draft of this text was in part supported by the research project “Spaces of Analytic Functions and Singular Integrals” of the University of St. Petersburg, RSF Grant 14-41-00010.

I am truly grateful to Éric Charpentier, my friend and colleague at the Université de Bordeaux, who – as with my preceding book (*Espaces de Hardy*, Belin, 2012) – sacrificed much of his time for a thorough and effective review of the complete text. Without his invaluable and generous aid the text would never have seen the light of day.

I warmly thank Albrecht Böttcher, the great expert in the Toeplitz domain, who took the time to read the manuscript and whose profound and nourishing comments aided me in polishing the text. I am also beholden to several of my

colleagues for *ad hoc* consultations on the subjects of the book, and especially to Anton Baranov of St. Petersburg for his remarks and suggestions concerning the editing.

I also address my cordial thanks to Gilles Godefroy (Institut de Mathématiques de Jussieu, Université Paris VI) who, at a delicate moment of the project, rescued it from a dead end brought on by circumstances beyond my control, by recommending the publisher Calvage et Mounet. I would of course also like to warmly thank the publisher (in particular Alain Debreil and Rachid Mneimné) for the painstaking and highly qualified attention they paid to the present manuscript, as well as for the friendly atmosphere that marked our collaboration on this occasion.

And finally, but foremost, I think of my large family – Pascale, Laure, Ivan, Jeanne, and Alekseï – who stoically supported my prolonged isolation while I struggled with my text, and who encouraged me in my moments of doubt.

Élancourt,
November 2016

Biographies

Bernhard Riemann	<i>page</i> 3	Igor Simonenko	<i>page</i> 113
Vito Volterra	5	Carl Runge	123
David Hilbert	7	George Birkhoff	170
Henri Poincaré	12	Nikolai Luzin	184
Otto Toeplitz	17	Norbert Wiener	192
Hermann Hankel	28	Eberhard Hopf	194
Marie Hankel(-Dippe)	28	Edmond Laguerre	200
Zeev Nehari	44	Josip Plemelj	212
Mark Krein	48	Constantin Carathéodory	232
Felix Berezin	60	Gábor Szegő	244
Leopold Kronecker	71	Hermann Weyl	272
Solomon Mikhlin	78	Israel Gelfand	342
Paul Halmos	83	Erik Ivar Fredholm	360
Gaston Julia	89	Frederick Atkinson	366
Augustin Cauchy	104	Fritz Noether	378
Israel Gohberg	108	Felix Hausdorff	381

Figures

Bernhard Riemann (Mathematisches Forschungsinstitut Oberwolfach gGmbH (MFO): https://opc.mfo.de/)	<i>page 2</i>
A self-caricature by Lewis Carroll (Culture Club/Hulton Archive/Getty Images)	3
Vito Volterra	5
David Hilbert (MFO)	7
Henri Poincaré	12
Otto Toeplitz (MFO)	17
Hermann Hankel	28
Figure 2.1 The function $\frac{t(\pi - t)}{ t }$.	40
Zeev Nehari (I Have a Photographic Memory by Paul R. Halmos; ©1987 American Mathematical Society)	44
Mark Grigorievich Krein (Ukraine Mathematical Society: www.imath.kiev.ua)	48
Felix A. Berezin (Courtesy of Elena Karpel, personal archive)	60
Leopold Kronecker (MacTutor History of Mathematics Archive: www-history.mcs.st-and.ac.uk)	71
Figure 2.2 The matrix A of the Hint of §2.3.10(a)	74
Solomon Mikhlin (Vladimir Maz'ya and Tatyana Shaposhnikova, uploaded by Daniele TampieriCC BY-SA 3.0: https://en.wikipedia.org/wiki/Solomon_Mikhlin)	78
Paul Halmos (MFO)	83
Gaston Julia in later years (MacTutor History of Mathematics Archive)	89
Gaston Julia in the French army during WWI (MacTutor History of Mathematics Archive)	90
Augustin Louis Cauchy (MacTutor History of Mathematics Archive)	104

Israel Gohberg (MacTutor History of Mathematics Archive)	108
Igor Simonenko (Ivleva.n.s/CC BY-SA 4.0: https://en.wikipedia.org/wiki/Igor_Simonenko)	113
Figure 3.1 The disk $D(\lambda, R)$, $\lambda \rightarrow \infty$	118
Figure 3.2 The disk $\overline{D}(1, d)$ of Lemma 3.3.4	120
Figure 3.3 The domain $\Omega(r, s, \alpha)$ of Lemma 3.3.4	120
Figure 3.4 The disk $\overline{D}(\lambda, R)$ and the domain $\Omega(r, s, \alpha)$ of Lemma 3.3.4	121
Carl Runge	123
George David Birkhoff (MFO)	170
Nikolai N. Luzin (MacTutor History of Mathematics Archive)	184
Norbert Wiener (Bettmann/Bettmann/Getty Images)	192
Eberhard Hopf (MacTutor History of Mathematics Archive)	194
Edmond N. Laguerre	200
Josip Plemelj (MacTutor History of Mathematics Archive)	212
Constantin Carathéodory (MacTutor History of Mathematics Archive)	232
Great Fire of Smyrna, 1922	233
Gàbor Szegő (MacTutor History of Mathematics Archive)	244
Figure 5.1 A banded Toeplitz matrix. The gray is a Toeplitz portion $\widehat{\varphi}(k - j)$, $0 \leq j, k \leq n$; the white is zero.	251
Hermann Weyl (MFO)	272
Israel Gelfand (Photo: Nick Romanenko, Copyright Rutgers, The State University of New Jersey)	342
Eric Ivar Fredholm (MacTutor History of Mathematics Archive)	360
Frederick V. Atkinson (MacTutor History of Mathematics Archive)	366
Fritz Noether (MFO)	378
Fritz Noether with Emmy Noether (MFO)	379
Oryol Prison, Russia	380
Felix Hausdorff	381