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Why Toeplitz–Hankel? Motivations and Panorama

Topics

- Four cornerstones of the theory of Toeplitz operators: the Riemann–Hilbert problem (RHP), the singular integral operators (SIO), the Wiener–Hopf operators (WHO), and last (but not least) the Toeplitz matrices and operators (TMO) (strictly speaking, compressions of multiplication operators).
- The founding contributions of Bernhard Riemann, David Hilbert, George Birkhoff, Otto Toeplitz, Gábor Szegő, Norbert Wiener, and Eberhard Hopf.
- The modern and post-modern periods of the theory.

Biographies Bernhard Riemann, Vito Volterra, David Hilbert, Henri Poincaré, Otto Toeplitz, Hermann and Marie Hankel.

1.1 Latent Maturation: The RHP and SIOs

The most ancient form of a "Toeplitz problem," which was not identified as such for a hundred years (!), is the Riemann, or Riemann–Hilbert, problem.

1.1.1 Nineteenth Century: Riemann and Volterra

Bernhard Riemann submitted his thesis in 1851 (under the direction of Gauss) and presented his inaugural dissertation entitled "Grundlagen für eine allgemeine Theorie der Funktionen einer veränderlich complexen Grösse" (also available in [Riemann, 1876]). Its principal value lay in its pioneering introduction of geometrical methods to the theory of functions, and in the objects that we now know under the names of Riemann surfaces, conformal mappings, and variational techniques. Moreover, among the 22 sections of this 43-page text (in today's format) there was a short Section 19 containing what is known (following Hilbert) as the ‘Riemann problem,” one of the cornerstones of the future theory of Toeplitz operators. In this Section 19 Riemann used almost no formulas, and he summed up himself, somewhat abstractly:

19. Überschlag über die hinreichenden und notwendigen Bedingungen zur Bestimmung einer Funktion complexen Argumenten innerhalb eines gegebenen Grössengebiet.

19. An outline of the necessary and sufficient conditions for the determination of a function of a complex variable in the interior of a given domain.

Bernhard Riemann (1826–1866) was a German mathematician, an ingenious creator whose contributions continue to fertilize mathematics 150 years after his passing. Riemann's ideas transformed complex analysis, geometry, and number theory, and also provided a strong impetus to real harmonic analysis and mathematical physics. Three of Riemann's four most influential works appeared as "qualification texts": his doctoral thesis (Göttingen, 1851, under the direction of Gauss), containing the theory of *Riemann surfaces* and conformal mappings, as well as what is now known as the *Riemann (boundary) problem* (RP); his *Habilitation* thesis (1853) devoted to trigonometric series (with the *Riemann integral* as a tool); and the famous *Habilitationsvortrag* (inaugural *Habilitation* conference, 1854) entitled "Über die Hypothesen, welche der Geometrie zu Grunde liegen," which stimulated profound interactions between geometry and physics, leading to Einstein's general theory of relativity. These three masterpieces were published posthumously. The fourth work was "Über die Anzahl der Primzahlen unter einer gegebenen Grösse" (1859), on the distribution of prime numbers, containing – among other subjects – the famous *Riemann hypothesis* (RH) on the zeros of the ζ function in the complex plane (this publication was also "obligatory," as Riemann was obliged to present an article to the Berlin Academy as a new corresponding member). Riemann's works became – and remain – absolutely fundamental to the mathematics and physics of the nineteenth to twenty-first centuries.

An astronomical number of publications are dedicated to the development of Riemann's ideas and results. For a presentation aimed at the general public, see, for example, *Bernhard Riemann 1826–1866: Turning Points in the Conception of Mathematics* by Detlef Laugwitz (Birkhäuser, 2008), or *Riemann: Le géomètre de la nature* by Rossana Tazzioli (Pour la Science, no. 12, 2002), or *Riemann* by Hans Freudenthal in the *Complete Dictionary of Scientific Biography* (2008). As noted in this last source, "Riemann's evolution was slow and his life short." He only produced 15 mathematical publications, but these rare works defined an epoch. Riemann's name is associated with almost a hundred important concepts, such as *Riemannian geometry*, the *Cauchy–Riemann equations*, *Riemann surfaces*, the *Riemann integral*, the *Riemann conformal mapping theorem*, the *Riemann–Hilbert method (problem)*, the *Riemann hypothesis*, the *Riemann–Lebesgue lemma*, the *Riemann sphere*, the *Riemann–Roch theorem*, etc. In particular, as is well known, Riemannian geometry was decisive in the creation of general relativity – and also in the inspiration of mathematician Charles Dodgson (better known by his literary pseudonym Lewis Carroll: see the sketched self-portrait) for his brilliant *Alice's Adventures in Wonderland* (1865) and *Through the Looking-Glass* (1871).

Riemann's career was slow and difficult right from the start, beginning even with the choice of mathematics as the subject of his studies: on his arrival in 1846 at the University of Göttingen (quite provincial at the time) he was forced by his father to enrol in the faculty of theology and was not able to switch to mathematics until he received his father's permission in 1847. There, Riemann flourished in the Gauss/Weber seminar, where mathematics and physics were intimately interlinked. It was only in 1854 that Riemann gave his first lectures in Göttingen. In 1857, he participated in a competition for a professor's position at the École Polytechnique of Zürich, but, because of his difficulties in oral expression, he lost to his colleague and friend Richard Dedekind (1831–1916). Riemann became a tenured professor at Göttingen only in 1859 (after the death of Per Gustav

Dirichlet, 1805–1859), and always suffered from a lack of students (his famous course on Abelian functions was only attended by three students, including Dedekind). A deterioration of his health (latent tuberculosis?) often forced him to seek refuge in Italy (1862–1866), and according to one of his students (in May 1861) Riemann was often "weak and fatigued" to the point where, at times, he "could not succeed in proving even the simplest results." The level of appreciation of his innovative results by his colleagues was also very low: in the 1860s, Enrico Betti (1823–1892), who had warmly welcomed Riemann to Italy, pointed out that "the works of Riemann are practically unknown to the scientific community" because of "the concision and the obscurity of the style of this eminent geometer"; Karl Weierstrass (1815–1897) underestimated the results of Riemann's thesis; in England, he remained "almost unknown," and "in France and Italy his works were often studied but not well understood" (according to Rossana Tazzioli (2002)). Effectively, Riemann's mathematical style was barely accessible and he often laid himself open to criticism. For example, one can mention the statement of the Riemann problem (neither made explicit, nor linked to the context: see §1.1.1, §4.6 for details), or the famous article "Über die Anzahl ..." (1859), "the most ingenuous and fruitful" according to Edmund Landau (1877–1938), which was written – according to Detlef Laugwitz – "in such a manner that it is not easy to understand how he reached his solution; we easily recognize, in this manner of erasing his traces, a student of Gauss!" These reactions of his colleagues of course contributed to the deterioration of Riemann's health: again in 1857, a prolific year for Riemann, according to Dedekind, he was "hypochondriacal to the extreme, mistrustful of others and of himself," and in 1863 (again according to Dedekind) he was already in a "sad state" with depressive tendencies. Riemann died while travelling in Italy on Lake Maggiore (1866) without even reaching his 40th birthday.

Riemann married Elise Koch in 1862; they had one daughter.

In fact, this Section 19 contains the very first statement of a now famous problem; the author did not relate it to the other questions in his thesis and did not follow it up. In modern terms (especially due to Hilbert, see §1.1.3 below), the problem is as follows:

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Excerpt
More Information

Given a bounded domain G in the complex plane $\mathbb{C}$ *and real functions a, b, c on the boundary* ∂G*, find a holomorphic function* $f \in \mathrm{Hol}(G)$, $f = u + iv$ ($u = \mathrm{Re}\, f$, $v = \mathrm{Im}\, f$) *such that its boundary values satisfy* $au + bv = c$ *on* ∂G.

Riemann saw this as a generalization of the *Dirichlet problem*: find a function u satisfying $\Delta u = 0$ in G (where $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the Laplacian in $\mathbb{R}^2$) and $u = c$ (a given function) on ∂G. A restatement of the Riemann problem (and the naming of it as such) was introduced by David Hilbert in the twentieth century (1905) in his famous courses (Göttingen, 1901–1910) on integral equations (see §1.1.3 below), and it is in this exact form that we will consider and resolve it in Chapter 4. At this stage, there is as yet no question of any Toeplitz matrices/operators.

We must add that before the decisive intervention of Hilbert, an article by Vito Volterra [1882] appeared in which the Riemann problem was clearly stated and discussed, but it went almost totally unnoticed. Volterra's name will appear again in our outline of the historic evolution of Toeplitz theory (see §1.3.2 below).

Vito Volterra (1860–1940) was an Italian mathematician, known for his research in partial differential equations, in real analysis, as well as in the theory of integral operators (Volterra operators) and integral-differential operators. He was among those who paved the way between the mathematics of the nineteenth and twentieth centuries. Moreover, Volterra was one of the creators (with Lotka, but independently) of the mathematical theory (the dynamics and the equilibrium) of communities of antagonistic species (predator/prey), and in particular, of the *Lotka–Volterra equation*. His theory of "functions of lines" (which Hadamard (1865–1963) named "functionals"), which dates to 1887, inaugurated modern functional analysis. Volterra authored the very first publication on the Riemann problem (see §1.1.1), as well as 235 other research articles. He is the author of half a dozen monographs, including his theory of the struggle for life [Volterra, 1931].

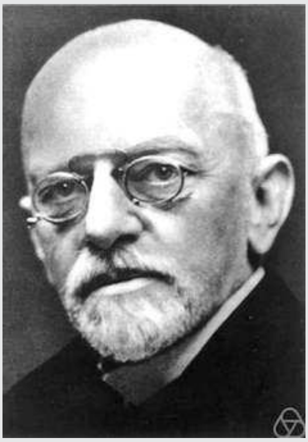
Volterra studied in Pisa, at the University and at the Scuola Normale Superiore, where he submitted his thesis in 1882 (under the direction of Enrico Betti), and then made a successful career as professor and chairman in Pisa, Turin and Rome. (His work in Turin was especially disturbed by a conflict with another great mathematician, Giuseppe Peano, known for his intolerance for the slightest weakness in the mathematical arguments of others, as well as for his project "Formulario Mathematico" (to code all of mathematics in symbols of logic and to teach it according to this source), which exasperated his colleagues.) Volterra is the only mathematician to have been invited four times to present a plenary conference at the ICM (1900, 1908, 1920, 1928). He became a member of several Academies (Royal Societies of London and Edinburgh, Accademia dei Lincei, and others) and received several distinctions; he founded a number of Italian research organizations (such as the Consiglio Nazionale delle Ricerche (1923) of which Volterra was the first president). Today, there are around a dozen mathematical objects bearing Volterra's name.

Volterra's career was greatly disrupted during the Fascist period in Italy. As one of the 12 Italian professors (out of a total of 1250) who refused to pledge allegiance to the Fascist government in 1931, he was fired from his position of professor and expelled from the Accademia dei Lincei; after the shameful "Manifesto della razza" (1938), Volterra (as well as his two sons who already had university positions) was expelled as a Jew from the Instituto Lombardo di Scienze e Lettere. He died of phlebitis at his home in Rome.

In 1900 Volterra married his cousin Virginia Almagià, and they had six children.

1.1.2 Twentieth Century: David Hilbert

David Hilbert played a multifaceted role in the evolution of Toeplitz operators: he made the first real advance for the Riemann problem (abbreviated RP), he launched "Problem 21" (of which the RP represents the principal part) in his celebrated 1900 list of 23 unresolved problems for the twentieth century, he indicated the principal technique for the solution of the RP (the singular integral operators – SIO), and finally (last but not least!) he suggested to Otto Toeplitz that he consider Laurent matrices/forms (which was to lead to the Toeplitz operators) as an illustration of his brand new spectral theory presented in his courses of 1901–1910 on integral operators.



David Hilbert (1862–1943) was a German mathematician whose works and personality exercised a decisive influence on all of the mathematics of the twentieth century. His contributions to number theory (via the theory of class fields in the famous *Zahlbericht*, "Report on numbers" (1897), written at the request of the German Mathematical Society), the axiomatization of geometry, integral equations and functional analysis, mathematical physics, the calculus of variations, and mathematical logic are fundamental. His speech to the second International Congress of Mathematicians (Paris, 1900), containing 23 unresolved problems across all domains, determined the development of mathematics for the following decades.

Several disciplines were quite simply created by the pen of David Hilbert, such as proof theory and metamathematics, or "spectral theory" (the name given by Hilbert to his theory of bounded self-adjoint operators; 25 years later, commenting on the quantum physics of Max Born, Werner Heisenberg, and Erwin Schrödinger, he remarked "I even called it 'spectral analysis' without any presentiment that it would later find applications to actual physical spectra").

His founding course on integral equations given at the University of Göttingen 1901–1908 (first published by his student Hermann Weyl in 1908, then in Hilbert's own book in 1912) contains what is known today as *Hilbert spaces*, the *Hilbert transform*, the *Hilbert inequality*, and the "general Riemann problem" (known, after Hilbert's intervention, as the *Riemann–Hilbert problem*: see §1.1.1 and Chapter 4 below). Hermann Weyl (student and then successor of Hilbert in his position, and himself also a master of analysis of the twentieth century) provided an overview and analysis of Hilbert's works by dividing them into five periods: see [Weyl, 1944].

Hilbert created the "Göttingen school," a community without precedent in the history of mathematics, with 69 students who had submitted a thesis under his direction, and which featured a plethora of key figures of twentieth-century mathematics, such as Otto Blumenthal, Felix Bernstein, Sergei Bernstein, Richard Courant, Alfréd Haar, Erich Hecke, Ernst Hellinger, Erhard Schmidt, Hugo Steinhaus, Hermann Weyl, and Adolf Hurwitz. Many others, without having formally been his doctoral students, spent long periods at Hilbert's seminar in Göttingen, including Emmy Noether, Harald Bohr, Max Born (Nobel Prize 1954), Emanuel Lasker (world chess champion, 1894 and 1921), Alonzo Church, John von Neumann, Otto Toeplitz, Hermann Weyl, Ernst Zermelo, and dozens of others.

Several dozen mathematical objects bear Hilbert's name (a few examples were given above); according to Constance Reid ([1970], page 216), "Like some mathematical Alexander, he had left his name written large across the map of mathematics."

Almost all of Hilbert's career was spent in Göttingen, apart from a short period in Königsberg (1892–1895 as a *Privatdozent*), where he married Käthe Jarosch (a long-time family friend) and had a son, Franz. After years of intense and triumphant work, Hilbert became gravely ill in 1925 (pernicious anaemia, incurable at the time) and was given only a few weeks to live (in fact, the first signs of weakness were already apparent in 1908). Hilbert was miraculously saved thanks to the efforts of his friends and colleagues (Richard Courant, Oliver Kellogg, George Birkhoff, and others) who organized a veritable human chain to bring a new experimental medicine from Harvard (at a time of crisis and cruel daily problems!). Hilbert retired in 1930, the year when his long-standing efforts finally led to the opening of the new Mathematical Institute. He continued to give his courses (one per year), but as he was already very weak, he often lost the thread of his reasoning or lost track of its final goal. It is no doubt this weakness that explains the sad fact that we find his signature on a collective letter of support for Adolf Hitler published before the referendum of 1934 (he who was already so hurt by the ethnic cleansing that had left his Institute bled dry).

Hilbert's life has given birth to a kind of mythology. His reaction to a new German currency in 1923 (in order to control the galloping inflation) was: "Impossible to resolve a problem by changing the name of an independent variable." His first article on invariants (containing among others the celebrated *Nullstellensatz*) was rejected by Paul Gordan, an expert with the *Mathematischen Annalen*, who remarked on the proof of (pure) existence of a finite number of generators: *Das ist nicht Mathematik. Das ist Theologie* ("That is not mathematics. That is theology").

Hilbert's credo was "A perfect formulation of a problem is already half of its solution" ([Reid, 1970], page 101). Then there is the famous *Wir müssen wissen, wir werden wissen* ("We must know, we will know"), the final motto of his retirement speech at Königsberg (1930), on the eve (!) of Kurt Gödel's announcement of his theorem of the incompleteness of the Zermelo–Fraenkel axiomatic system. (The phrase itself seems to be an allusion to a memorable German slogan, *Wir müssen siegen, und wir werden siegen* ("We must win, and we will win"), used in particular by Kaiser Wilhelm II in August 1914, in war propaganda on thousands of postcards and medals, in war songs, etc., and which probably originated from the *Zwinger Saga* (1517–1524), a medieval text about the siege of the imperial city of Goslar. The Germans were not the only ones – think of *Venceremos*, etc.)

In his courses, first published in *Nachrichten der Köningliche Gesellschaft der Wissenschaften zu Göttingen* between 1904 and 1910, starting with "Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen" [Hilbert, 1904], later published as a book [Hilbert, 1912], Hilbert generalized the RP and provided a formulation in terms of singular integral operators (SIO). In particular, with the use of the Riemann conformal mapping theorem (proved in his inaugural dissertation mentioned above, up to a small vague detail on the applicability of the "Dirichlet principle"), Riemann's question stated in §1.1.1 can be reduced (if G is simply connected) to $G = \mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$. Then, since $u = (f + \overline{f})/2$, $v = (f - \overline{f})/2i$, and $\overline{f}|\partial\mathbb{D}$ is the boundary value of a function $f_-(z) = \overline{f}(1/\overline{z})$ holomorphic in $\mathbb{C} \setminus \overline{\mathbb{D}}$, the problem can be reformulated in the following manner.

Given functions A, B, C on the circle $\partial\mathbb{D} = \mathbb{T} = \{z \in \mathbb{C} \colon |z| = 1\}$, find two holomorphic functions $f_+ \in \mathrm{Hol}(\mathbb{D})$ and $f_- \in \mathrm{Hol}(\mathbb{C} \setminus \overline{\mathbb{D}})$ such that their boundary values satisfy

$$Af_+ + Bf_- = C$$

on $\mathbb{T}$ *(and* $f_-(\infty) = 0$*).*

Under the hypothesis that A is never zero on $\mathbb{T}$ (and under some rather vague conditions on the regularity of A, B, C: it seems to be $A, B, C \in C^2$), Hilbert reduced the question to a singular integral equation (in fact by identifying $f_\pm$ with projections $P_\pm F$ of a function on $\mathbb{T}$: see Chapter 4 for details) that belonged to a family of equations already studied (in the same course),

$$a(s)\varphi(s) + \int_{[-\pi,\pi]} K(s,t)\varphi(t)\,dt = \psi(s),$$

where K is a kernel holomorphic outside the diagonal $s = t$, on which it has a simple pole, and can be written in the form

$$K(s,t) = b(s)\,\mathrm{ctg}\,\frac{s-t}{2} + N(s,t)$$

with $N \in L^2([-\pi,\pi]^2)$ (see Chapter 4 for details). Hilbert thus inferred that for $n = \mathrm{wind}(B/A) = 0$ ("winding number," the Cauchy index of the curve $B/A(\mathbb{T})$: see Definitions 3.1.2 below) a solution exists and is unique; for $n \neq 0$, it is necessary to impose $|n|$ supplementary conditions, or else obtain n independent solutions – a conclusion which anticipated the concept of the index of an operator (introduced by Noether only in 1921: see Appendix E). Finally, this technique led to a complete solution of the RP, presented in Chapter 4 below.

In the same book, Hilbert ([1912], Chapter X) applied the results about the RP to the *problem of monodromy groups* of linear ordinary differential equations (ODEs), also proposed by Riemann. Initially, the problem concerns ODEs of order n in the complex plane $\mathbb{C}$; by a change of notation, such an equation can be reduced to a system of n equations of first degree, i.e.

$$dy(z)/dz = A(z)y(z)$$

where $y = (y_1, \ldots, y_n)^{\mathrm{col}}$ and A is an $n \times n$ matrix-valued function. The system is said to be *Fuchsian* (after Lazarus Fuchs, a German mathematician, 1833–1902) if A is holomorphic in $\mathbb{C}$ except at a finite set of points z_j where it has simple poles. The poles of A provoke eventual branches (logarithmic) of y along an analytic extension on a given closed curve $\gamma\colon [0,1] \to \Omega$ in $\Omega = \mathbb{C} \setminus \{z_j\}$, which leads to the equality $y \circ \gamma(1) = C(y \circ \gamma(0))$ where $C = C_\gamma$ is an invertible matrix with constant elements. Clearly, if $\gamma = \gamma_1 \cdot \gamma_2$ is a composite path, then $C_\gamma = C_{\gamma_1} C_{\gamma_2}$; the image $\gamma \mapsto C_\gamma$ of the fundamental group of Ω in the group of invertible matrices $\mathbb{C}^n \to \mathbb{C}^n$ is called the *monodromy group* of the equation. Problem 21 in Hilbert's famous list of problems (1900) questions whether an arbitrary group of $n \times n$ matrices can be the monodromy group of a certain Fuchsian system.

Hilbert [1912] showed how, for $n = 2$, the monodromy problem can be reduced to a vectorial Riemann problem $f_- = af_+$ ($f_\pm = (f^1_\pm, f^2_\pm)^{\mathrm{col}}$, a a constant 2×2 matrix, $\det(a) \neq 0$), and then resolved this RP. (In the English-speaking engineering literature, this presentation of the RP is known as the "barrier problem.") Hilbert's appreciation and treatment of the RP (with the aid of the SIOs), as well as its links with the problem of monodromy groups, attracted a strong focus to the RP and gave birth to powerful techniques with a broad spectrum of applications: see §3.5 and 4.6 for a description of some of this research. Impressed by this approach, Émile Picard [1927] gave it the nickname