EQUIVALENTS OF THE RIEMANN HYPOTHESIS

Volume One: Arithmetic Equivalents

The Riemann hypothesis (RH) is perhaps the most important outstanding problem in mathematics. This two-volume text presents the main known equivalents to RH using analytic and computational methods. The books are gentle on the reader with definitions repeated, proofs split into logical sections, and graphical descriptions of the relations between different results. They also include extensive tables, supplementary computational tools, and open problems suitable for research. Accompanying software is free to download.

These books will interest mathematicians who wish to update their knowledge, graduate and senior undergraduate students seeking accessible research problems in number theory, and others who want to explore and extend results computationally. Each volume can be read independently.

Volume 1 presents classical and modern arithmetic equivalents to RH, with some analytic methods. Volume 2 covers equivalences with a strong analytic orientation, supported by an extensive set of appendices containing fully developed proofs.

Encyclopedia of Mathematics and Its Applications

This series is devoted to significant topics or themes that have wide application in mathematics or mathematical science and for which a detailed development of the abstract theory is less important than a thorough and concrete exploration of the implications and applications.

Books in the Encyclopedia of Mathematics and Its Applications cover their subjects comprehensively. Less important results may be summarized as exercises at the ends of chapters. For technicalities, readers can be referred to the bibliography, which is expected to be comprehensive. As a result, volumes are encyclopedic references or manageable guides to major subjects.
Encyclopedia of Mathematics and Its Applications

All the titles listed below can be obtained from good booksellers or from Cambridge University Press. For a complete series listing visit www.cambridge.org/mathematics.

119 M. Deza and M. Dutour Sikirić  Geometry of Chemical Graphs

131 T. G. Faticoni  Modules over Endomorphism Rings

133 H. Morimoto  Stochastic Control and Mathematical Modeling

135 B. Courcelle and J. Engelfriet  Graph Structure and Monadic Second-Order Logic

137 M. Baake and U. Grimm  Aperiodic Order II: Representation Theory and the Zelmanov Approach

139 M. Baake and U. Grimm  Matrices and Graphs in Geometry

141 G. Schmidt  Relational Mathematics

143 R. Schneider  Convex Bodies: The Brunn–Minkowski Theory (Second Edition)

145 M. Cabrera García and A. Rodríguez Palacios  Non-Associative Normed Algebras I: The Vidav–Palmer and Gelfand–Naimark Theorems

147 C. F. Dunkl and Y. Xu  Orthogonal Polynomials of Several Variables (Second Edition)

149 L. Berlyand, A. G. Kolpakov and A. Novikov  Introduction to the Network Approximation Method for Materials Modeling

151 M. Baake and U. Grimm  Aperiodic Order I: A Mathematical Invitation

153 D. Hofmann, G. J. Seal and W. Tholen (eds.)  Monoidal Topology

155 C. F. Dunkl and Y. Xu  Orthogonal Polynomials and Continued Fractions

157 T. Mora  Solving Polynomial Equation Systems IV: Buchberger Theory and Beyond

159 V. Berthé and M. Rigo (eds.)  Combinatorics, Automata and Number Theory

161 M. Ghergu and S. D. Taliaferro  Isolated Singularities in Partial Differential Equations

163 W. T. Broughan  Equivalents of the Riemann Hypothesis I: Arithmetic Equivalents

165 K. Broughan  Equivalents of the Riemann Hypothesis II: Analytic Equivalents

167 M. Baake and U. Grimm  Aperiodic Order II: Representation Theory and the Zelmanov Approach
Equivalents of the Riemann Hypothesis
Volume One: Arithmetic Equivalents

KEVIN BROUGHAN
University of Waikato, New Zealand
Dedicated to Jackie, Jude and Beck
RH is a precise statement, and in one sense what it means is clear, but what it is connected with, what it implies, where it comes from, can be very unobvious.

Martin Huxley
Contents for Volume One

Contents for Volume Two
List of Illustrations xiv
List of Tables xvi
Preface for Volume One xvii
List of Acknowledgements xxi

1 Introduction
1.1 Chapter Summary 1
1.2 Early History 1
1.3 Volume One Summary 8
1.4 Notation 12
1.5 Background Reading 13
1.6 Unsolved Problems 14

2 The Riemann Zeta Function
2.1 Introduction 15
2.2 Basic Properties 16
2.3 Zero-Free Regions 21
2.4 Landau’s Zero-Free Region 25
2.5 Zero-Free Regions Summary 29
2.6 The Product Over Zeta Zeros 30
2.7 Unsolved Problems 39

3 Estimates
3.1 Introduction 40
3.2 Constructing Tables of Bounds for $\psi(x)$ 41
3.3 Exact Verification Using Computation 51
3.4 Estimates for $\theta(x)$ 54
3.5 More Estimates 65
3.6 Unsolved Problems 67
4 Classical Equivalences

4.1 Introduction 68
4.2 The Prime Number Theorem and Its RH Equivalences 69
4.3 Oscillation Theorems 81
4.4 Errors in Arithmetic Sums 88
4.5 Unsolved Problems 93

5 Euler’s Totient Function

5.1 Introduction 94
5.2 Estimates for Euler’s Function \( \varphi(n) \) 98
5.3 Preliminary Results With RH True 110
5.4 Further Results With RH True 123
5.5 Preliminary Results With RH False 130
5.6 Nicolas’ First Theorem 135
5.7 Nicolas’ Second Theorem 137
5.8 Unsolved Problems 142

6 A Variety of Abundant Numbers

6.1 Introduction 144
6.2 Superabundant Numbers 147
6.3 Colossally Abundant Numbers 153
6.4 Estimates for \( x^2(\varepsilon) \) 161
6.5 Unsolved Problems 163

7 Robin’s Theorem

7.1 Introduction 165
7.2 Ramanujan’s Theorem Assuming RH 169
7.3 Preliminary Lemmas With RH True 174
7.4 Bounding \( \prod_{p \leq x} (1 - p^{-2}) \) From Above With RH True 180
7.5 Bounding \( \log \log N \) From Below With RH True 184
7.6 Proof of Robin’s Theorem With RH True 186
7.7 An Unconditional Bound for \( \sigma(n)/n \) 188
7.8 Bounding \( \log \log N \) From Above Without RH 190
7.9 A Lower Bound for \( \sigma(n)/n \) With RH False 191
7.10 Lagarias’ Formulation of Robin’s Criterion 193
7.11 Unconditional Results for Lagarias’ Formulation 196
7.12 Unitary Divisor Sums 197
7.13 Unsolved Problems 198

8 Numbers That Do Not Satisfy Robin’s Inequality

8.1 Introduction 200
8.2 Hardy–Ramanujan Numbers 202
8.3 Integers Not Divisible by the Fifth Power of Any Prime 208
Contents for Volume One

8.4 Integers Not Divisible by the Seventh Power of Any Prime 211
8.5 Integers Not Divisible by the 11th Power of Any Prime 214
8.6 Unsolved Problems 217

9 Left, Right and Extremely Abundant Numbers 218
9.1 Introduction 218
9.2 Grönwall’s Theorem 220
9.3 Further Preliminary Results 223
9.4 Riemann Hypothesis Equivalences 225
9.5 Comparing Colossally and Left Abundant Numbers 232
9.6 Extremely Abundant Numbers 235
9.7 Unsolved Problems 235

10 Other Equivalents to the Riemann Hypothesis 236
10.1 Introduction 236
10.2 Shapiro’s Criterion 239
10.3 Farey Fractions 241
10.4 Redheffer Matrix 247
10.5 Divisibility Graph 250
10.6 Dirichlet Eta Function 252
10.7 The Derivative of $\zeta(s)$ 253
10.8 A Zeta-Related Inequality 256
10.9 The Real Part of the Logarithmic Derivative of $\xi(s)$ 259
10.10 The Order of Elements of the Symmetric Group 271
10.11 Hilbert–Pólya Conjecture 282
10.12 Epilogue 285
10.13 Unsolved Problems 286

Appendix A Tables 287
A.1 Extremely Abundant Numbers 287
A.2 Small Numbers Not Satisfying Robin’s Inequality 288
A.3 Superabundant Numbers 289
A.4 Colossally Abundant Numbers 290
A.5 Primes to Make Colossally Abundant Numbers 291
A.6 Small Numbers Satisfying Nicolas’ Reversed Inequality 292
A.7 Heights of Integers 293
A.8 Maximum Order of an Element of the Symmetric Group 293

Appendix B RHpack Mini-Manual 294
B.1 Introduction 294
B.2 RHpack Functions 296

References 313
Index 321
Contents for Volume Two

Contents for Volume One
List of Illustrations xiv
List of Tables xvi
Preface for Volume Two xvii
List of Acknowledgements xxi

1 Introduction 1
  1.1 Why This Study? 1
  1.2 Summary of Volume Two 2
  1.3 How to Read This Book 7

2 Series Equivalents 8
  2.1 Introduction 8
  2.2 The Riesz Function 10
  2.3 Additional Properties of the Riesz Function 14
  2.4 The Series of Hardy and Littlewood 15
  2.5 A General Theorem for a Class of Entire Functions 16
  2.6 Further Work 22

3 Banach and Hilbert Space Methods 23
  3.1 Introduction 23
  3.2 Preliminary Definitions and Results 25
  3.3 Beurling’s Theorem 29
  3.4 Recent Developments 35

4 The Riemann Xi Function 37
  4.1 Introduction 37
  4.2 Preliminary Results 40
  4.3 Monotonicity of |ξ(s)| 49
Contents for Volume Two

4.4 Positive Even Derivatives 51
4.5 Li’s Equivalence 54
4.6 More Recent Results 59

5 The De Bruijn–Newman Constant 62
5.1 Introduction 62
5.2 Preliminary Definitions and Results 66
5.3 A Region for $\Xi(z)$ With Only Real Zeros 69
5.4 The Existence of $\Lambda$ 77
5.5 Improved Lower Bounds for $\Lambda$ 77
  5.5.1 Lehmer’s Phenomenon 78
  5.5.2 The Differential Equation Satisfied by $H(t,z)$ 81
  5.5.3 Finding a Lower Bound for $\Lambda$ Using Lehmer Pairs 87
5.6 Further Work 92

6 Orthogonal Polynomials 93
6.1 Introduction 93
6.2 Definitions 94
6.3 Orthogonal Polynomial Properties 96
6.4 Moments 99
6.5 Quasi-Analytic Functions 104
6.6 Carleman’s Inequality 106
6.7 Riemann Zeta Function Application 113
6.8 Recent Work 116

7 Cyclotomic Polynomials 117
7.1 Introduction 117
7.2 Definitions 118
7.3 Preliminary Results 119
7.4 Riemann Hypothesis Equivalences 124
7.5 Further Work 126

8 Integral Equations 127
8.1 Introduction 127
8.2 Preliminary Results 129
8.3 The Method of Sekatskii, Beltramelli and Merlini 133
8.4 Salem’s Equation 139
8.5 Levinson’s Equivalence 142

9 Weil’s Explicit Formula, Inequality and Conjectures 150
9.1 Introduction 150
9.2 Definitions 152
9.3 Preliminary Results 152
9.4 Weil’s Explicit Formula 154
## Contents for Volume Two

9.5 Weil’s Inequality 159
9.6 Bombieri’s Variational Approach to RH 166
9.7 Introduction to the Weil Conjectures 173
9.8 History of the Weil Conjectures 174
9.9 Finite Fields 176
9.10 The Weil Conjectures for Varieties 178
9.11 Elliptic Curves 178
9.12 Weil Conjectures for Elliptic Curves – Preliminary Results 182
9.13 Proof of the Weil Conjectures for Elliptic Curves 186
9.14 General Curves Over \( \mathbb{F}_q \) and Applications 188
9.15 Return to the Explicit Formula 190
9.16 Weil’s Commentary on his 1952 and 1972 Papers 192

10 Discrete Measures 193
10.1 Introduction 193
10.2 Definitions 194
10.3 Preliminary Results 195
10.4 A Mellin-Style Transform 197
10.5 Verjovsky’s Theorems 200
10.6 Historical Development of Non-Euclidean Geometry 206
10.7 The Hyperbolic Upper Half Plane \( \mathbb{H} \) 208
10.8 The Groups PSL(2, \( \mathbb{R} \)) and PSL(2, \( \mathbb{Z} \)) 209
10.9 Eisenstein Series 211
10.10 Zagier’s Horocycle Equivalence 216
10.11 Additional Results 219

11 Hermitian Forms 221
11.1 Introduction 221
11.2 Definitions 223
11.3 Distributions 226
11.4 Positive Definite 228
11.5 The Restriction to \( C(a) \) for All \( a > 0 \) 231
11.6 Properties of \( K(a) \) and \( \hat{K(a)} \) 236
11.7 Matrix Elements 242
11.8 An Explicit Example With \( a = \log \sqrt{2} \) 247
11.9 Lemmas for Yoshida’s Main Theorem 258
11.10 Hermitian Forms Lemma 260
11.11 Yoshida’s Main Theorem 269
11.12 The Restriction to \( K(a) \) for All \( a > 0 \) 270

12 Dirichlet \( L \)-Functions 274
12.1 Introduction 274
12.2 Definitions 277
Contents for Volume Two

12.3 Properties of $L(s,\chi)$
12.4 The Non-Vanishing of $L(1,\chi)$
12.5 Zero-Free Regions and Siegel Zeros
12.6 Preliminary Results for Titchmarsh’s Criterion
12.7 Titchmarsh’s GRH Equivalence
12.8 Preliminary Results for Gallagher’s Theorem
12.9 Gallagher’s Theorems
12.10 Applications of Gallagher’s Theorems
12.11 The Bombieri–Vinogradov Theorem
12.12 Applications of Bombieri–Vinogradov’s Theorem
12.13 Generalizations and Developments for Bombieri–Vinogradov
12.14 Conjectures

13 Smooth Numbers
13.1 Introduction
13.2 The Dickman Function
13.3 Preliminary Lemmas for Hildebrand’s Equivalence
13.4 Riemann Hypothesis Equivalence
13.5 Further Work

14 Epilogue

Appendix A Convergence of Series
Appendix B Complex Function Theory
Appendix C The Riemann–Stieltjes Integral
Appendix D The Lebesgue Integral on $\mathbb{R}$
Appendix E The Fourier Transform
Appendix F The Laplace Transform
Appendix G The Mellin Transform
Appendix H The Gamma Function
Appendix I The Riemann Zeta Function
Appendix J Banach and Hilbert Spaces
Appendix K Miscellaneous Background Results
Appendix L GRHpack Mini-Manual
  L.1 Introduction
    L.1.1 Installation
    L.1.2 About This Mini-Manual
  L.2 GRHpack Functions

References
Index
Illustrations

1.1 Euclid (about 325–265 BCE). page 2
1.2 L. Euler, 1707–1783. 3
1.3 J. C. F. Gauss, 1777–1855. 4
1.4 J. L. Dirichlet, 1805–1859. 4
1.5 B. Riemann, 1826–1866. 5
1.6 David Hilbert, 1862–1943. 7
1.7 Some relationships between chapters. 12
2.1 The contours of $|\zeta(s)|$. 17
2.2 The flow $\dot{s} = \zeta(s)$. 18
2.3 The flow $\dot{s} = \xi(s)$. 19
2.4 The function $-S(t)$ for $1000 \leq t \leq 1040$. 24
2.5 The function $T(t)$ on [950, 1060] with a phantom zero at $t = 1010$. 25
3.1 The values of $\epsilon_b$ for $10 \leq b \leq 100$ and $1 \leq m \leq 5$. 52
3.2 The values of $\epsilon_b$ for $100 \leq b \leq 1000$ and $3 \leq m \leq 7$. 52
3.3 The values of $\epsilon_b$ for $10^3 \leq b \leq 10^4$ and $3 \leq m \leq 7$. 53
3.4 A plot of $b$, where $x = e^b$, vs $\log e(x)$ for $10^3 \leq b \leq 10^4$. 60
3.5 A plot of $b$, where $x = e^b$, vs $\log e(x)$ for $10^5 \leq b \leq 10^6$. 60
4.1 J. E. Littlewood (1885–1977). 87
4.2 The function $M(x)/\sqrt{x}$ for $1 \leq x \leq 10^5$. 91
5.1 Jean-Louis Nicolas. 95
5.2 Some relationships between results in Chapter 5. 98
5.3 Some relationships between results in Chapter 5. 99
5.4 The function $f(x)$. 113
5.5 The function $H(x)$. 116
5.6 The function $F_{1/2}(x)$. 120
5.7 Some relationships between results in Chapter 5. 126
5.8 The relation $c(n) \leq c(N_k)$ for $k = 3$. 129
5.9 Proportion of solutions to $e^\gamma \log \log n < n / \varphi(n)$ for $1 \leq n \leq 10^5$. 137
5.10 The function $\Delta_k$ for $1 \leq k \leq 4000$. 137
List of Illustrations

5.11 The sequence $c(n)$. 138
5.12 The sequence $c(N_k)$ for $120,568 \leq k \leq 10^6$. 141
6.1 Paul Erdős, 1913–1996. 145
6.2 The functions $F(3, \alpha)$ and $F(2, \alpha)$ showing $F(3, \alpha) < F(2, \alpha)$. 154
6.3 The function $n \to \sigma(n)/n^{1+\epsilon}$. 157
6.4 The function $n \to \sigma(n)/n^{1+0.1}$. 158
7.1 The values of $\sigma(n)/(n \log \log n)$ for $5041 \leq n \leq 6041$. 168
7.2 Timothy Trudgian. 169
7.3 Relationships between some of the results in Chapter 7. 170
7.4 Further relationships between some of the results in Chapter 7. 171
7.5 Additional relationships between some of the results in Chapter 7. 171
7.6 Further relationships between some of the results in Chapter 7. 172
7.7 Jeffrey Lagarias. 194
7.8 The values of $\sigma(n)/(H_n + \exp(H_n) \log(H_n))$ for $2 \leq n \leq 1000$. 196
8.1 YoungJu Choie. 201
9.1 Dependences between some results in this chapter. 220
9.2 Further dependences between some results in this chapter. 221
9.3 More dependences between some results in this chapter. 221
10.1 The Farey sums over $\sqrt{n}$ with $1 \leq n \leq 100$. 248
10.2 Individual terms in the Farey sum for $n = 101$. 248
10.3 The divisibility graph for $n = 6$. 250
10.4 The first 300 imaginary coordinate gaps between zeros of $\zeta(s)$. 267
10.5 The function $S(T)$ on the domain $[400, 450]$. 268
10.6 Values of the function $h(n)$ excluding prime powers. 274
10.7 Values of the function $\log g(n)/\sqrt{n \log n}$ for $3 \leq n \leq 1000$. 280
Tables

2.1 Some improvements for Backlund’s zero counting estimate. page 20
2.2 Zeros on the critical line and corresponding heights. 22
2.3 Examples for finding values of R. 29
3.1 Upper bounds used for k_m(0), 1 ≤ m ≤ 28. 50
3.2 Values of b, m and ε_b. 50
3.3 More values of b, m and ε_b. 51
3.4 Values of b = log x, and rounded log ε(x) for R = 8 and R = 18. 59
3.5 Larger values of b and log ε(x). 59
6.1 Indexing for the first eight colossally abundant numbers. 158
7.1 Evaluation of η(α, a, b), given α, a and b. 185
8.1 Values of n_1(t) and N_{n_1(t)}. 217
10.1 The first 20 Riemann zeros and gaps. 266
A.1 The first seven extremely abundant numbers. 287
A.2 Numbers n ≤ 5041 with cr(n)/n ≥ e^γ loglogn. 288
A.3 The first 31 superabundant numbers. 289
A.4 The first 30 colossally abundant numbers. 290
A.5 Initial values of h(n); compare with item A008475 in OEIS. 293
A.6 Initial values of g(n); compare with item A000793 in OEIS. 293
Preface

Why have these two volumes on equivalences to the Riemann hypothesis been written? Many would say that the Riemann hypothesis (RH) is the most noteworthy problem in all of mathematics. This is not only because of its relationship to the distribution of prime numbers, the fundamental building blocks of arithmetic, but also because there exist a host of related conjectures that will be resolved if RH is proved to be true and which will be proved to be false if the converse is demonstrated. These are the RH equivalences. The lists of equivalent conjectures have continued to grow ever since the hypothesis was first enunciated, over 150 years ago.

The many attacks on RH that have been reported, the numerous failed attempts, and the efforts of the many whose work has remained obscure, have underlined the problem’s singular nature. So too has its mythology. The great English number theorist, Godfrey Hardy, wrote a postcard to Harald Bohr while returning to Cambridge from Denmark in rough weather that read: “Have proof of RH. Postcard too short for proof.” He didn’t believe in a God, but was certain he would not be allowed to drown with his name associated with an infamous missing proof. David Hilbert, the renowned German mathematician, was once asked, “If you were to die and be revived after five hundred years, what would you then do?” Hilbert replied that he would ask “Has someone proved the Riemann hypothesis?” More recently, towards the end of the twentieth century, Enrico Bombieri, an Italian mathematician at the Institute for Advanced Study, Princeton, issued a joke email announcing the solution of RH by a young physicist, on 1 April of course!

There are several ways in which the truth of the hypothesis has been supported but not proved. These have included increasing the finite range of values $T > 0$ such that the imaginary part of all complex zeros of $\zeta(s)$ up to $T$ all have real part $\frac{1}{2}$ [68], increasing the lower bound for the proportion
Preface

of zeros that are on the critical line $\Re s = \frac{1}{2}$ [40], and increasing the size of the region in the complex plane where $\zeta(s)$ can be proved to be non-zero [63].

This volume includes a detailed account of some recent work that takes a different approach. It is based on inequalities involving some simple accessible arithmetic functions. Broadly outlined RH implies that an inequality is true for all integers, or all integers sufficiently large, or all integers of a particular type. If RH fails, there is an integer in the given range, or of the given type, for which the inequality fails, so the truth of the inequality is equivalent to the truth of RH. Progress under this approach is made whenever the nature of any counterexample is shown to be more restricted than previously demonstrated. The reader may also wish to consult the introductions to Chapters 3 to 7 for further details.

The relatively recent work depends critically on a range of explicit estimates for arithmetic functions. These have been derived using greater computing power than was available in the 1940s and 1960s when these sorts of estimates were first published. In many cases the details are included, along with simplified presentations.

Also included are a range of other equivalences to RH, some by now classical. The more recent work depends on these classical equivalences for both the results and techniques, so it is useful to set both out explicitly. It also shows how some equivalences are more fundamental than others. This is not to suggest new equivalences are easy consequences of older established ones, even though this may be true in some cases.

The aim of these volumes is to give graduate students and number theory researchers easy access to these methods and results in order that they might build on them. To this end, complete proofs have been included wherever possible, so readers might judge for themselves their depth and crucial steps. To provide context, a range of additional equivalences has been included in this volume, some of which are arithmetical and some more analytic. An intuitive background for some of the functions employed is also included in the form of graphical representations. Numerical calculations have been reworked, and values different from those found in the literature have often been arrived at.

To aid the reader, definitions are often repeated and major steps in proofs are numbered to give a clear indication of the main parts and allow for easy proof internal referencing. When possible, errors in the literature have been corrected. Where a proof has not been verified, either because this author was not able to fill gaps in the argument, or because it was incorrect, it has not been included. There is a website for errata and corrigenda, and readers are encouraged to communicate with the author in this regard at kab@waikato.ac.nz. The website is linked to the author’s homepage: www.math.waikato.ac.nz/~kab.
Preface xix

Also linked to this website is a suite of Mathematica™ programs, called RHpack, related to the material in this volume, which is available for free download. Instructions on how to download the software are given in Appendix B.

The two volumes are distinct, with a small amount of overlap. This volume, Volume One, has an arithmetic orientation, with some analytic methods, especially those relying on the manipulation of inequalities. The equivalences found here are the Môbius mu estimate of Littlewood, the explicit $\psi(x)$ function estimate of Schoenfeld, the Liouville $\lambda(n)$ limit criterion of Landau, two Euler totient function criteria of Nicolas, the sum of divisors inequality of Ramanujan and Robin and its reformulation by Lagarias, the criterion of Caveney, Nicolas and Sondow based on so-called “extraordinary numbers”, the criterion of Nazardonyavi and Yakubovich based on extremely abundant numbers, the estimate of Shapiro that uses the integral of $\psi(x)$, the Franel–Landau Farey fraction criterion, the divisibility matrix criterion of Redheffer, the Levinson–Montgomery criterion that uses counts of the zeros of the derivative $\zeta'(s)$, the inequality of Spira relating values of zeta at $s$ and $1 - s$, the self-adjoint operator criteria of Hilbert and Pólya, and the criterion of Lagarias and Garunkstis based on the real part of the logarithmic derivative of $\xi(s)$. In addition, Volume One has criteria based on the divisibility matrix of Redheffer and a closely related graph, the Dirichlet eta function, and an estimate for the size of the maximum order of an element of the symmetric group.

The Appendix includes tables of the numbers that appear in some of the equivalences and a mini-manual for RHpack.

Volume Two [32] contains equivalences with a strong analytic orientation. To support these, there is an extensive set of appendices containing fully developed proofs. The equivalences set out are named Amoroso, Hardy–Littlewood, Báez-Duarte, Beurling, Bombieri, Bombieri–Lagarias, de Bruijn–Newman, Cardon–Roberts, Hildebrand, Levinson, Li, Riesz, Sekatskii–Beltraminelli–Merlini, Salem, Sondow–Dumitrescu, Verjovsky, Weil, Yoshida and Zagier. For summary details, see the Preface for Volume Two and Chapter 1 of Volume Two. In addition, Bombieri’s proof of Weil’s explicit formula, a discussion of the Weil conjectures and a proof of the conjectures for elliptic curves are included.

In the case of the general Riemann hypothesis (GRH) for Dirichlet $L$-functions, in Volume Two the Titchmarsh criterion is given, as well as proofs of the Bombieri–Vinogradov and Gallagher theorems and a range of their applications. There is a small supporting Mathematica package, GRHpack.

The set of appendices for Volume Two gives comprehensive statements and proofs of the special results that are needed to derive the equivalences in that volume. In addition, there is a GRHpack mini-manual.
Preface

A note concerning the cover figure. This represents integral paths for the flow $\dot{s} = \zeta(s)$ in a small region of the upper complex plane, rotated and reflected in $\sigma = 0$. It was produced using an interactive program written by the author and Francis Kuo in Java®. It includes three critical zeros and three trivial zeros.

Many people have assisted with the development and production of these volumes. Without their help and support, the work would not have been possible, and certainly not completed in a reasonable period of time. They include Sir Michael Berry, Enrico Bombieri, Jude Broughan, George Csordas, Daniel Delbourgo, Tomás García Ferrari, Pat Gallagher, Adolf Hildebrand, Geoff Holmes, Stephen Joe, Jeff Lagarias, Wayne Smith, Tim Trudgian, John Turner and Michael Wilson. The support of the University of Waikato and especially its Faculty of Computing and Mathematical Sciences and Department of Mathematics and Statistics has been absolutely essential. Cambridge University Press has also provided much encouragement and support, especially Roger Astley and Clare Dennison. Last, but not least, I am grateful for my family’s belief in me and support of my work.

Kevin Broughan

December 2016
Acknowledgements

The author gratefully acknowledges the following sources and/or permissions for the non-exclusive use of copyrighted material.

Euclid: Figure 1.1, being an excerpt from Raphael’s fresco “School of Athens”, Visions of America/Superstock/Getty Images.

L. Euler: Figure 1.2, Apic/Hulton Archive/Getty Images.

C. F. Gauss: Figure 1.3, Bettmann/Getty Images.

J. L. Dirichlet: Figure 1.4, Stringer/Hulton Archive/Getty Images.

B. Riemann: Figure 1.5, Author: Konrad Jacobs. Source: Archives of the Mathematisches Forschungsinstitut Oberwolfach.

D. Hilbert: Figure 1.6, Ullstein bild/Getty Images.


Xi flow: Figure 2.3, being figure 1, p. 1274, of K. A. Broughan, The holomorphic flow of Riemann’s function ξ(s), Nonlinearity 18 (2005), 1289–1294. Copyright © IOP Publishing and London Mathematical Society. Permission of the American Mathematical Society. All rights reserved.

J. E. Littlewood: Figure 4.1, used by permission of the Master and Fellows of Trinity College Cambridge.

J.-L. Nicolas: Figure 5.1, used by permission of J.-L. Nicolas.

P. Erdős: Figure 6.1, Author: Kay Piene. Source: Ragni Piene and Archives of the Mathematisches Forschungsinstitut Oberwolfach.

T. S. Trudgian: Figure 7.2, used by permission of T. S. Trudgian.

J. C. Lagarias: Figure 7.7, used by permission of J. C. Lagarias.

Y. Choie: Figure 8.1, used by permission of Y. Choie.

Section 8.5 was derived from a previously published article: K. A. Broughan and T. Trudgian, Robin’s inequality for 11-free integers, Integers 15 (2015), A12 (5pp.).

xxi