

EQUIVALENTS OF THE RIEMANN HYPOTHESIS

Volume One: Arithmetic Equivalents

The Riemann hypothesis (RH) is perhaps the most important outstanding problem in mathematics. This two-volume text presents the main known equivalents to RH using analytic and computational methods. The books are gentle on the reader with definitions repeated, proofs split into logical sections, and graphical descriptions of the relations between different results. They also include extensive tables, supplementary computational tools, and open problems suitable for research. Accompanying software is free to download.

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ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

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Volume One: Arithmetic Equivalents

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University of Waikato, New Zealand



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Dedicated to Jackie, Jude and Beck

RH is a precise statement, and in one sense what it means is clear, but what it is connected with, what it implies, where it comes from, can be very unobvious.

Martin Huxley

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Preface

Why have these two volumes on equivalences to the Riemann hypothesis been written? Many would say that the Riemann hypothesis (RH) is the most noteworthy problem in all of mathematics. This is not only because of its relationship to the distribution of prime numbers, the fundamental building blocks of arithmetic, but also because there exist a host of related conjectures that will be resolved if RH is proved to be true and which will be proved to be false if the converse is demonstrated. These are the RH equivalences. The lists of equivalent conjectures have continued to grow ever since the hypothesis was first enunciated, over 150 years ago.

The many attacks on RH that have been reported, the numerous failed attempts, and the efforts of the many whose work has remained obscure, have underlined the problem's singular nature. So too has its mythology. The great English number theorist, Godfrey Hardy, wrote a postcard to Harald Bohr while returning to Cambridge from Denmark in rough weather that read: "Have proof of RH. Postcard too short for proof." He didn't believe in a God, but was certain he would not be allowed to drown with his name associated with an infamous missing proof. David Hilbert, the renowned German mathematician, was once asked, "If you were to die and be revived after five hundred years, what would you then do?" Hilbert replied that he would ask "Has someone proved the Riemann hypothesis?" More recently, towards the end of the twentieth century, Enrico Bombieri, an Italian mathematician at the Institute for Advanced Study, Princeton, issued a joke email announcing the solution of RH by a young physicist, on 1 April of course!

There are several ways in which the truth of the hypothesis has been supported but not proved. These have included increasing the finite range of values $T > 0$ such that the imaginary part of all complex zeros of $\zeta(s)$ up to T all have real part $\frac{1}{2}$ [68], increasing the lower bound for the proportion

of zeros that are on the critical line $\Re s = \frac{1}{2}$ [40], and increasing the size of the region in the complex plane where $\zeta(s)$ can be proved to be non-zero [63].

This volume includes a detailed account of some recent work that takes a different approach. It is based on inequalities involving some simple accessible arithmetic functions. Broadly outlined RH implies that an inequality is true for all integers, or all integers sufficiently large, or all integers of a particular type. If RH fails, there is an integer in the given range, or of the given type, for which the inequality fails, so the truth of the inequality is equivalent to the truth of RH. Progress under this approach is made whenever the nature of any counterexample is shown to be more restricted than previously demonstrated. The reader may also wish to consult the introductions to Chapters 3 to 7 for further details.

The relatively recent work depends critically on a range of explicit estimates for arithmetic functions. These have been derived using greater computing power than was available in the 1940s and 1960s when these sorts of estimates were first published. In many cases the details are included, along with simplified presentations.

Also included are a range of other equivalences to RH, some by now classical. The more recent work depends on these classical equivalences for both the results and techniques, so it is useful to set both out explicitly. It also shows how some equivalences are more fundamental than others. This is not to suggest new equivalences are easy consequences of older established ones, even though this may be true in some cases.

The aim of these volumes is to give graduate students and number theory researchers easy access to these methods and results in order that they might build on them. To this end, complete proofs have been included wherever possible, so readers might judge for themselves their depth and crucial steps. To provide context, a range of additional equivalences has been included in this volume, some of which are arithmetical and some more analytic. An intuitive background for some of the functions employed is also included in the form of graphical representations. Numerical calculations have been reworked, and values different from those found in the literature have often been arrived at.

To aid the reader, definitions are often repeated and major steps in proofs are numbered to give a clear indication of the main parts and allow for easy proof internal referencing. When possible, errors in the literature have been corrected. Where a proof has not been verified, either because this author was not able to fill gaps in the argument, or because it was incorrect, it has not been included. There is a website for errata and corrigenda, and readers are encouraged to communicate with the author in this regard at kab@waikato.ac.nz. The website is linked to the author's homepage: www.math.waikato.ac.nz/~kab.

Also linked to this website is a suite of *Mathematica*TM programs, called RHpack, related to the material in this volume, which is available for free download. Instructions on how to download the software are given in Appendix B.

The two volumes are distinct, with a small amount of overlap. This volume, Volume One, has an arithmetic orientation, with some analytic methods, especially those relying on the manipulation of inequalities. The equivalences found here are the Möbius mu estimate of Littlewood, the explicit $\psi(x)$ function estimate of Schoenfeld, the Liouville $\lambda(n)$ limit criterion of Landau, two Euler totient function criteria of Nicolas, the sum of divisors inequality of Ramanujan and Robin and its reformulation by Lagarias, the criterion of Caveney, Nicolas and Sondow based on so-called “extraordinary numbers”, the criterion of Nazardonyavi and Yakubovich based on extremely abundant numbers, the estimate of Shapiro that uses the integral of $\psi(x)$, the Franel–Landau Farey fraction criterion, the divisibility matrix criterion of Redheffer, the Levinson–Montgomery criterion that uses counts of the zeros of the derivative $\zeta'(s)$, the inequality of Spira relating values of zeta at s and $1-s$, the self-adjoint operator criteria of Hilbert and Pólya, and the criterion of Lagarias and Garunkstis based on the real part of the logarithmic derivative of $\xi(s)$. In addition, Volume One has criteria based on the divisibility matrix of Redheffer and a closely related graph, the Dirichlet eta function, and an estimate for the size of the maximum order of an element of the symmetric group.

The Appendix includes tables of the numbers that appear in some of the equivalences and a mini-manual for RHpack.

Volume Two [32] contains equivalences with a strong analytic orientation. To support these, there is an extensive set of appendices containing fully developed proofs. The equivalences set out are named Amoroso, Hardy–Littlewood, Báez-Duarte, Beurling, Bombieri, Bombieri–Lagarias, de Bruijn–Newman, Cardon–Roberts, Hildebrand, Levinson, Li, Riesz, Sekatskii–Beltraminelli–Merlini, Salem, Sondow–Dumitrescu, Verjovsky, Weil, Yoshida and Zagier. For summary details, see the Preface for Volume Two and Chapter 1 of Volume Two. In addition, Bombieri’s proof of Weil’s explicit formula, a discussion of the Weil conjectures and a proof of the conjectures for elliptic curves are included.

In the case of the general Riemann hypothesis (GRH) for Dirichlet L -functions, in Volume Two the Titchmarsh criterion is given, as well as proofs of the Bombieri–Vinogradov and Gallagher theorems and a range of their applications. There is a small supporting *Mathematica* package, GRHpack.

The set of appendices for Volume Two gives comprehensive statements and proofs of the special results that are needed to derive the equivalences in that volume. In addition, there is a GRHpack mini-manual.

A note concerning the cover figure. This represents integral paths for the flow $\dot{s} = \zeta(s)$ in a small region of the upper complex plane, rotated and reflected in $\sigma = 0$. It was produced using an interactive program written by the author and Francis Kuo in Java[®]. It includes three critical zeros and three trivial zeros.

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