EQUIVALENTS OF THE RIEMANN HYPOTHESIS

Volume One: Arithmetic Equivalents

The Riemann hypothesis (RH) is perhaps the most important outstanding problem in mathematics. This two-volume text presents the main known equivalents to RH using analytic and computational methods. The books are gentle on the reader with definitions repeated, proofs split into logical sections, and graphical descriptions of the relations between different results. They also include extensive tables, supplementary computational tools, and open problems suitable for research. Accompanying software is free to download.

These books will interest mathematicians who wish to update their knowledge, graduate and senior undergraduate students seeking accessible research problems in number theory, and others who want to explore and extend results computationally. Each volume can be read independently.

Volume 1 presents classical and modern arithmetic equivalents to RH, with some analytic methods. Volume 2 covers equivalences with a strong analytic orientation, supported by an extensive set of appendices containing fully developed proofs.

Encyclopedia of Mathematics and Its Applications

This series is devoted to significant topics or themes that have wide application in mathematics or mathematical science and for which a detailed development of the abstract theory is less important than a thorough and concrete exploration of the implications and applications.

Books in the **Encyclopedia of Mathematics and Its Applications** cover their subjects comprehensively. Less important results may be summarized as exercises at the ends of chapters. For technicalities, readers can be referred to the bibliography, which is expected to be comprehensive. As a result, volumes are encyclopedic references or manageable guides to major subjects.

ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

All the titles listed below can be obtained from good booksellers or from Cambridge University Press. For a complete series listing visit

www.cambridge.org/mathematics.

119 M. Deza and M. Dutour Sikirić *Geometry of Chemical Graphs*
120 T. Nishiura *Absolute Measurable Spaces*
121 M. Prest *Purity, Spectra and Localisation*
122 S. Khrushchev *Orthogonal Polynomials and Continued Fractions*
123 H. Nagamochi and T. Ibaraki *Algorithmic Aspects of Graph Connectivity*
124 F. W. King *Hilbert Transforms I*
125 F. W. King *Hilbert Transforms II*
126 O. Calin and D.-C. Chang *Sub-Riemannian Geometry*
127 M. Grabisch *et al. Aggregation Functions*
128 L. W. Beineke and R. J. Wilson (eds.) with J. L. Gross and T. W. Tucker *Topics in Topological Graph Theory*
129 J. Berstel, D. Perrin and C. Reutenauer *Codes and Automata*
130 T. G. Faticoni *Modules over Endomorphism Rings*
131 H. Morimoto *Stochastic Control and Mathematical Modeling*
132 G. Schmidt *Relational Mathematics*
133 P. Kornerup and D. W. Matula *Finite Precision Number Systems and Arithmetic*
134 Y. Crama and P. L. Hammer (eds.) *Boolean Models and Methods in Mathematics, Computer Science, and Engineering*
135 V. Berthé and M. Rigo (eds.) *Combinatorics, Automata and Number Theory*
136 A. Kristály, V. D. Rădulescu and C. Varga *Variational Principles in Mathematical Physics, Geometry, and Economics*
137 J. Berstel and C. Reutenauer *Noncommutative Rational Series with Applications*
138 B. Courcelle and J. Engelfriet *Graph Structure and Monadic Second-Order Logic*
139 M. Fiedler *Matrices and Graphs in Geometry*
140 N. Vakil *Real Analysis through Modern Infinitesimals*
141 R. B. Paris *Hadamard Expansions and Hyperasymptotic Evaluation*
142 Y. Crama and P. L. Hammer *Boolean Functions*
143 A. Arapostathis, V. S. Borkar and M. K. Ghosh *Ergodic Control of Diffusion Processes*
144 N. Caspard, B. Leclerc and B. Monjardet *Finite Ordered Sets*
145 D. Z. Arov and H. Dym *Bitangential Direct and Inverse Problems for Systems of Integral and Differential Equations*
146 G. Dassios *Ellipsoidal Harmonics*
147 L. W. Beineke and R. J. Wilson (eds.) with O. R. Oellermann *Topics in Structural Graph Theory*
148 L. Berlyand, A. G. Kolpakov and A. Novikov *Introduction to the Network Approximation Method for Materials Modeling*
149 M. Baake and U. Grimm *Aperiodic Order I: A Mathematical Invitation*
150 J. Borwein *et al. Lattice Sums Then and Now*
151 R. Schneider *Convex Bodies: The Brunn–Minkowski Theory (Second Edition)*
152 G. Da Prato and J. Zabczyk *Stochastic Equations in Infinite Dimensions (Second Edition)*
153 D. Hofmann, G. J. Seal and W. Tholen (eds.) *Monoidal Topology*
154 M. Cabrera García and Á. Rodríguez Palacios *Non-Associative Normed Algebras I: The Vidav–Palmer and Gelfand–Naimark Theorems*
155 C. F. Dunkl and Y. Xu *Orthogonal Polynomials of Several Variables (Second Edition)*
156 L. W. Beineke and R. J. Wilson (eds.) with B. Toft *Topics in Chromatic Graph Theory*
157 T. Mora *Solving Polynomial Equation Systems III: Algebraic Solving*
158 T. Mora *Solving Polynomial Equation Systems IV: Buchberger Theory and Beyond*
159 V. Berthé and M. Rigo (eds.) *Combinatorics, Words and Symbolic Dynamics*
160 B. Rubin *Introduction to Radon Transforms: With Elements of Fractional Calculus and Harmonic Analysis*
161 M. Ghergu and S. D. Taliaferro *Isolated Singularities in Partial Differential Inequalities*
162 G. Molica Bisci, V. Radulescu and R. Servadei *Variational Methods for Nonlocal Fractional Problems*
163 S. Wagon *The Banach–Tarski Paradox (Second Edition)*
164 K. Broughan *Equivalents of the Riemann Hypothesis I: Arithmetic Equivalents*
165 K. Broughan *Equivalents of the Riemann Hypothesis II: Analytic Equivalents*
166 M. Baake and U. Grimm *Aperiodic Order II: Representation Theory and the Zelmanov Approach*

ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

Equivalents of the Riemann Hypothesis

Volume One: Arithmetic Equivalents

KEVIN BROUGHAN
University of Waikato, New Zealand

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

4843/24, 2nd Floor, Ansari Road, Daryaganj, Delhi – 110002, India

79 Anson Road, #06-04/06, Singapore 079906

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of
education, learning, and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9781107197046
DOI: 10.1017/9781108178228

First published 2017

Printed in the United Kingdom by Clays, St Ives plc

A catalogue record for this publication is available from the British Library.

Library of Congress Cataloging-in-Publication Data
Names: Broughan, Kevin A. (Kevin Alfred), 1943– author.
Title: Equivalents of the Riemann hypothesis / Kevin Broughan,
University of Waikato, New Zealand.
Description: Cambridge : Cambridge University Press, 2017– |
Series: Encyclopedia of mathematics and its applications ; 164 |
Includes bibliographical references and index. Contents: volume 1. Arithmetic equivalents
Identifiers: LCCN 2017034308 | ISBN 9781107197046 (hardback : alk. paper : v. 1)
Subjects: LCSH: Riemann hypothesis.
Classification: LCC QA246 .B745 2017 |
DDC 512.7/3–dc23 LC record available at https://lccn.loc.gov/2017034308

ISBN – 2 Volume Set 978-1-108-29078-4 Hardback
ISBN – Volume 1 978-1-107-19704-6 Hardback
ISBN – Volume 2 978-1-107-19712-1 Hardback

Dedicated to Jackie, Jude and Beck

RH is a precise statement, and in one sense what it means is clear, but what it is connected with, what it implies, where it comes from, can be very un-obvious.

Martin Huxley

Contents for Volume One

Contents for Volume Two

Illustrations

Tables

Preface

Why have these two volumes on equivalences to the Riemann hypothesis been written? Many would say that the Riemann hypothesis (RH) is the most noteworthy problem in all of mathematics. This is not only because of its relationship to the distribution of prime numbers, the fundamental building blocks of arithmetic, but also because there exist a host of related conjectures that will be resolved if RH is proved to be true and which will be proved to be false if the converse is demonstrated. These are the RH equivalences. The lists of equivalent conjectures have continued to grow ever since the hypothesis was first enunciated, over 150 years ago.

The many attacks on RH that have been reported, the numerous failed attempts, and the efforts of the many whose work has remained obscure, have underlined the problem's singular nature. So too has its mythology. The great English number theorist, Godfrey Hardy, wrote a postcard to Harald Bohr while returning to Cambridge from Denmark in rough weather that read: "Have proof of RH. Postcard too short for proof." He didn't believe in a God, but was certain he would not be allowed to drown with his name associated with an infamous missing proof. David Hilbert, the renowned German mathematician, was once asked, "If you were to die and be revived after five hundred years, what would you then do?" Hilbert replied that he would ask "Has someone proved the Riemann hypothesis?" More recently, towards the end of the twentieth century, Enrico Bombieri, an Italian mathematician at the Institute for Advanced Study, Princeton, issued a joke email announcing the solution of RH by a young physicist, on 1 April of course!

There are several ways in which the truth of the hypothesis has been supported but not proved. These have included increasing the finite range of values $T > 0$ such that the imaginary part of all complex zeros of $\zeta(s)$ up to T all have real part $\frac{1}{2}$ [68], increasing the lower bound for the proportion of zeros that are on the critical line $\Re s = \frac{1}{2}$ [40], and increasing the size of the region in the complex plane where $\zeta(s)$ can be proved to be non-zero [63].

This volume includes a detailed account of some recent work that takes a different approach. It is based on inequalities involving some simple accessible arithmetic functions. Broadly outlined RH implies that an inequality is true for all integers, or all integers sufficiently large, or all integers of a particular type. If RH fails, there is an integer in the given range, or of the given type, for which the inequality fails, so the truth of the inequality is equivalent to the truth of RH. Progress under this approach is made whenever the nature of any counterexample is shown to be more restricted than previously demonstrated. The reader may also wish to consult the introductions to Chapters 3 to 7 for further details.

The relatively recent work depends critically on a range of explicit estimates for arithmetic functions. These have been derived using greater computing power than was available in the 1940s and 1960s when these sorts of estimates were first published. In many cases the details are included, along with simplified presentations.

Also included are a range of other equivalences to RH, some by now classical. The more recent work depends on these classical equivalences for both the results and techniques, so it is useful to set both out explicitly. It also shows how some equivalences are more fundamental than others. This is not to suggest new equivalences are easy consequences of older established ones, even though this may be true in some cases.

The aim of these volumes is to give graduate students and number theory researchers easy access to these methods and results in order that they might build on them. To this end, complete proofs have been included wherever possible, so readers might judge for themselves their depth and crucial steps. To provide context, a range of additional equivalences has been included in this volume, some of which are arithmetical and some more analytic. An intuitive background for some of the functions employed is also included in the form of graphical representations. Numerical calculations have been reworked, and values different from those found in the literature have often been arrived at.

To aid the reader, definitions are often repeated and major steps in proofs are numbered to give a clear indication of the main parts and allow for easy proof internal referencing. When possible, errors in the literature have been corrected. Where a proof has not been verified, either because this author was not able to fill gaps in the argument, or because it was incorrect, it has not been included. There is a website for errata and corrigenda, and readers are encouraged to communicate with the author in this regard at `kab@waikato.ac.nz`. The website is linked to the author's homepage: `www.math.waikato.ac.nz/~kab`.

Also linked to this website is a suite of *Mathematica*™ programs, called RHpack, related to the material in this volume, which is available for free download. Instructions on how to download the software are given in Appendix B.

The two volumes are distinct, with a small amount of overlap. This volume, Volume One, has an arithmetic orientation, with some analytic methods, especially those relying on the manipulation of inequalities. The equivalences found here are the Möbius mu estimate of Littlewood, the explicit $\psi(x)$ function estimate of Schoenfeld, the Liouville $\lambda(n)$ limit criterion of Landau, two Euler totient function criteria of Nicolas, the sum of divisors inequality of Ramanujan and Robin and its reformulation by Lagarias, the criterion of Caveney, Nicolas and Sondow based on so-called "extraordinary numbers", the criterion of Nazardonyavi and Yakubovich based on extremely abundant numbers, the estimate of Shapiro that uses the integral of $\psi(x)$, the Franel–Landau Farey fraction criterion, the divisibility matrix criterion of Redheffer, the Levinson–Montgomery criterion that uses counts of the zeros of the derivative $\zeta'(s)$, the inequality of Spira relating values of zeta at s and $1-s$, the self-adjoint operator criteria of Hilbert and Pólya, and the criterion of Lagarias and Garunkstis based on the real part of the logarithmic derivative of $\xi(s)$. In addition, Volume One has criteria based on the divisibility matrix of Redheffer and a closely related graph, the Dirichlet eta function, and an estimate for the size of the maximum order of an element of the symmetric group.

The Appendix includes tables of the numbers that appear in some of the equivalences and a mini-manual for RHpack.

Volume Two [32] contains equivalences with a strong analytic orientation. To support these, there is an extensive set of appendices containing fully developed proofs. The equivalences set out are named Amoroso, Hardy–Littlewood, Báez-Duarte, Beurling, Bombieri, Bombieri–Lagarias, de Bruijn–Newman, Cardon–Roberts, Hildebrand, Levinson, Li, Riesz, Sekatskii–Beltraminelli–Merlini, Salem, Sondow–Dumitrescu, Verjovsky, Weil, Yoshida and Zagier. For summary details, see the Preface for Volume Two and Chapter 1 of Volume Two. In addition, Bombieri's proof of Weil's explicit formula, a discussion of the Weil conjectures and a proof of the conjectures for elliptic curves are included.

In the case of the general Riemann hypothesis (GRH) for Dirichlet L-functions, in Volume Two the Titchmarsh criterion is given, as well as proofs of the Bombieri–Vinogradov and Gallagher theorems and a range of their applications. There is a small supporting *Mathematica* package, GRHpack.

The set of appendices for Volume Two gives comprehensive statements and proofs of the special results that are needed to derive the equivalences in that volume. In addition, there is a GRHpack mini-manual.

A note concerning the cover figure. This represents integral paths for the flow $\dot{s} = \zeta(s)$ in a small region of the upper complex plane, rotated and reflected in $\sigma = 0$. It was produced using an interactive program written by the author and Francis Kuo in Java®. It includes three critical zeros and three trivial zeros.

Many people have assisted with the development and production of these volumes. Without their help and support, the work would not have been possible, and certainly not completed in a reasonable period of time. They include Sir Michael Berry, Enrico Bombieri, Jude Broughan, George Csordas, Daniel Delbourgo, Tomás Garcia Ferrari, Pat Gallagher, Adolf Hildebrand, Geoff Holmes, Stephen Joe, Jeff Lagarias, Wayne Smith, Tim Trudgian, John Turner and Michael Wilson. The support of the University of Waikato and especially its Faculty of Computing and Mathematical Sciences and Department of Mathematics and Statistics has been absolutely essential. Cambridge University Press has also provided much encouragement and support, especially Roger Astley and Clare Dennison. Last, but not least, I am grateful for my family's belief in me and support of my work.

Kevin Broughan
December 2016

Acknowledgements

The author gratefully acknowledges the following sources and/or permissions for the non-exclusive use of copyrighted material.

Euclid: Figure 1.1, being an excerpt from Raphael's fresco "School of Athens", Visions of America/Superstock/Getty Images.

L. Euler: Figure 1.2, Apic/Hulton Archive/Getty Images.

C. F. Gauss: Figure 1.3, Bettmann/Getty Images.

J. L. Dirichlet: Figure 1.4, Stringer/Hulton Archive/Getty Images.

B. Riemann: Figure 1.5, Author: Konrad Jacobs. Source: Archives of the Mathematisches Forschungsinstitut Oberwolfach.

D. Hilbert: Figure 1.6, Ullstein bild/Getty Images.

Zeta flow: Figure 2.2, being figure 2, p. 990, of K. A. Broughan and A. R. Barnett, *The holomorphic flow of the Riemann zeta function*, Mathematics of Computation **73** (2004), 287–1004. Permission of the American Mathematical Society.

Xi flow: Figure 2.3, being figure 1, p. 1274, of K. A. Broughan, *The holomorphic flow of Riemann's function* $\xi(s)$, Nonlinearity **18** (2005), 1289–1294. Copyright © IOP Publishing and London Mathematical Society. Permission of the American Mathematical Society. All rights reserved.

J. E. Littlewood: Figure 4.1, used by permission of the Master and Fellows of Trinity College Cambridge.

J.-L. Nicolas: Figure 5.1, used by permission of J.-L. Nicolas.

P. Erdős: Figure 6.1, Author: Kay Piene. Source: Ragni Piene and Archives of the Mathematisches Forschungsinstitut Oberwolfach.

T. S. Trudgian: Figure 7.2, used by permission of T. S. Trudgian.

J. C. Lagarias: Figure 7.7, used by permission of J. C. Lagarias.

Y. Choie: Figure 8.1, used by permission of Y. Choie.

Section 8.5 was derived from a previously published article: K. A. Broughan and T. Trudgian, *Robin's inequality for 11-free integers*, Integers **15** (2015), A12 (5pp.).