1
Introduction

1.1 Chapter Summary

This chapter is discursive. It begins with an overview of the early history of the Riemann hypothesis (RH) and the evolution of ideas relating to the Ramanujan–Robin inequality. Then in Section 1.3 there is a summary of the contents of the entire volume, first in brief and then in more detail. The section also describes the tables in Appendix A and the software RHpack in Appendix B.

There is a section on notational conventions and special notations, most of which are quite standard. A guide to the reader and two problems complete the chapter.

1.2 Early History

Here the main players in the evolution of the Riemann hypothesis are noted: Euclid, Euler, Gauss, Dirichlet and, last but not least, Riemann himself. The first is Euclid of Alexandria (Figure 1.1) who lived around 300 BCE. His *Elements* includes a proof that there are an infinite number of primes, and that they are the fundamental building blocks of numbers, through the unique factorization of integers. How are the primes distributed? On the face of it, the only pattern appears to be that all are odd, except 2. Do they appear completely at random, or are they uniformly distributed in some sense?

Don Zagier gives a good description of this random/uniform dichotomy: "The first fact is the prime numbers belong to the most arbitrary and ornery objects studied by mathematicians: they grow like weeds among the natural numbers, seeming to obey no other law than that of chance, and nobody can predict where the next one will sprout."

"The second fact is even more astonishing, for it states just the opposite: that the prime numbers exhibit stunning regularity, that there are laws governing their behaviour, and that they obey these laws with almost military precision."

Figure 1.1 Euclid (about 325–265 BCE).

Euler (1707–1783; Figure 1.2) was an amazing mathematician, who wrote a text on calculus, including the first treatment of trigonometric functions, and invented many parts of mathematics that are important today, including the calculus of variations, graph theory and divergent series. In number theory, quadratic reciprocity and Euler products are two of his many contributions, as well as his extensive work on the zeta function. In fact, Euler gave birth to the Riemann zeta function, writing down its definition for the first time. He studied the following sums and product:

$$\zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6},$$
$$\zeta(4) = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots = \frac{\pi^4}{90},$$
$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots,$$
$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}.$$

For Euler, the variable s was an integer, and the product over all primes. He tried in vain to find a closed expression for $\zeta(3)$ having the same style as that for $\zeta(2)$ and $\zeta(4)$.

Figure 1.2 L. Euler, 1707–1783.

The next main player was Gauss (1777–1855; Figure 1.3), although he does not appear to have used the zeta function. He was the giant of nineteenth-century mathematics and physics. For example, he used calculus to compute correctly the position of Ceres after it had passed behind the Sun. Although he spent most of his life in his observatory in Göttingen in Germany, he contributed to statistics, non-Euclidean geometry, curvature, geodesy, electromagnetism and complex numbers, as well as to number theory, for which he had an abiding passion.

Gauss was given a book of log tables, at around the age of 14, that included a table of primes. He extended the table, counting the number of primes up to a real positive variable x, now called $\pi(x)$. He considered the average number of primes in each interval of numbers $[1,2,3,\dots,N]$, and in this way arrived at his prime number conjecture, an inspired guess that was proved about 100 years later:

$$\pi(x) \sim \frac{x}{\log x} \quad \text{which means} \quad \frac{\pi(x)}{x/\log x} \to 1, \quad x \to \infty.$$

It was this conjecture that inspired many mathematicians during the nineteenth century.

Dirichlet (1805–1859; Figure 1.4) was a fine teacher and had a great influence on Bernhard Riemann, who attended his number theory lectures in Berlin during 1847–1849. According to Jacobi, Dirichlet was not only creative, but knew how to make a robust proof: “Only Dirichlet, not I, nor

Figure 1.3 J. C. F. Gauss, 1777–1855.

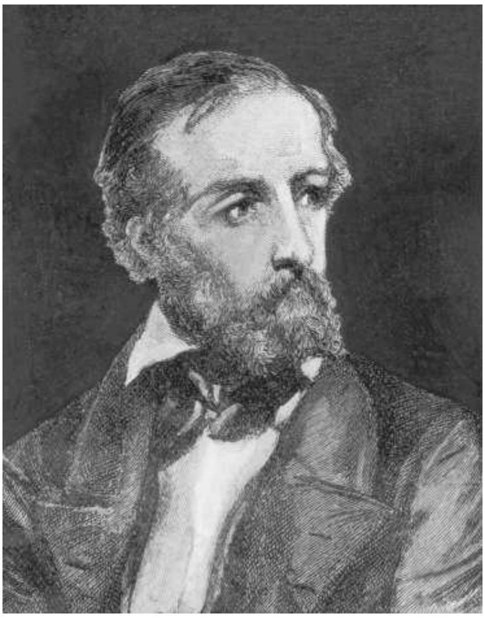

Figure 1.4 J. L. Dirichlet, 1805–1859.

Cauchy, not Gauss, knows what a perfectly rigorous proof is, but we learn it only from him. When Gauss says he has proved something it is very likely. When Cauchy says it, it's a fifty-fifty bet. When Dirichlet says it, it is certain." Dirichlet showed that there were an infinite number of primes in arithmetic progressions whenever the step and initial value are coprime. He used groups, characters and zeta functions. He was a wonderfully inspiring lecturer, and joined Riemann in Göttingen in 1855.

According to Felix Klein: "Riemann was bound to Dirichlet by the strong inner sympathy of a like mode of thought. Dirichlet loved to make things clear to himself with an intuitive underpinning. Along with this he would give acute, logical analyses of foundational questions and avoid long computations as much as possible." His manner suited Riemann, who adopted it and worked in many ways according to Dirichlet's methods.

So we come to the central character in this account, Bernhard Riemann (1826–1866; Figure 1.5). Riemann was impoverished for most of his life. He was shy and withdrawn, sickly and a hypochondriac. Other than the two years in Berlin, he spent most of his working life in the still very beautiful walled German town of Göttingen, but he changed mathematics forever. As well as numbers, formulae and concepts, the idea of mathematical spaces and of the

Figure 1.5 B. Riemann, 1826–1866.

relationships between them can be traced back to Riemann. He went beyond Dirichlet by supporting his profound ideas with extensive calculations and manipulations.

Riemann contributed to real analysis (the Riemann integral), complex analysis, (Cauchy–Riemann equations), potential theory and geometry (Riemannian manifolds). He richly deserves to be called a genius and a great mathematician.

An eight-page paper Riemann published in the Notices of the Berlin Academy in 1859, titled *On the number of primes less than a given magnitude* [140], contained an outline of a possible proof of Gauss's prime number conjecture. It started from ideas of Cauchy and Dirichlet. For example, he extended the domain of $\zeta(s)$ to the whole of the complex plane other than the point $s = 1$. The paper did not contain proofs, but radically changed analytic number theory. It took 30 years for mathematicians to begin to appreciate what Riemann's ideas really meant, and that his assertions were provable. Over 70 years later the analytic and computational underpinnings for the paper, through Siegel and his exploration of Riemann's hand-written notes, became clear. Riemann states in the paper what is now known as the Riemann hypothesis, or RH. Here is a translation of what he wrote:

One finds in fact about this many zeros of the zeta function within these bounds on the critical line, and it is **very likely that all of the zeros are on the critical line**. *One would of course like to have a rigorous proof of this, but I have put aside the search for such a proof after some fleeting vain attempts, because it is not necessary for the immediate objective of my investigation.* (For information about the term "critical line" see Section 2.2 (7).)

Indeed it was shown later that to complete the proof of the prime number theorem, the main objective of the paper, one did not need to prove the RH, only to show that the zero-free region for the zeta function includes all of the line $\Re s = 1$.

At the 1900 International Congress of Mathematicians, ICM1900, held in Paris, the great mathematician David Hilbert (Figure 1.6) gave an address in which he listed the 23 most outstanding problems of the day he considered worthy of the efforts of mathematicians, and fundamental for the further advancement of the subject. Problem number eight was the "Riemann hypothesis", which conjectured that the complex zeros of $\zeta(s)$ all had real part $\frac{1}{2}$.

The final event in this brief history, and significant for the work reported in this volume, is one of the consequences of assuming the hypothesis is true. In 1914, soon after the commencement of World War I when resources where scarce, the great Indian mathematician Srinivasa Ramanujan published an article in the Proceedings of the London Mathematical Society titled *Highly composite numbers*. At over 50 pages long it must have represented a considerable publishing challenge. As it turned out, there was a large amount

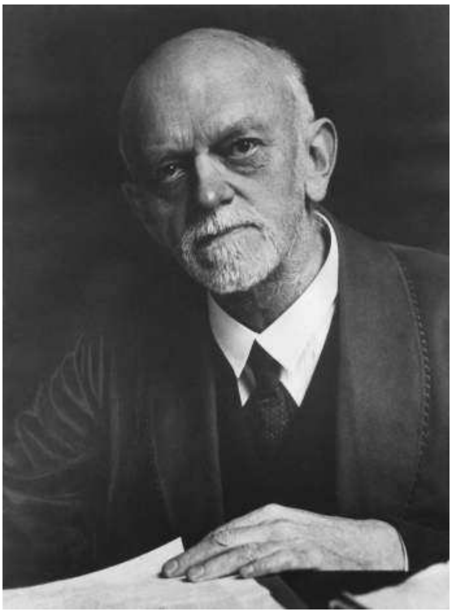

Figure 1.6 David Hilbert, 1862–1943.

of additional material on related topics written by Ramanujan which was not included in the published work. This was discovered in more recent times, and a typeset version with notes was eventually published in 1997. It includes an inequality, derived by Ramanujan, who assumed the Riemann hypothesis in this case, relating to the sum-of-divisors of an integer arithmetic function $\sigma(n)$. This can be stated as follows: for all $n \in \mathbb{N}$ sufficiently large we have

$$\frac{\sigma(n)}{n} < e^{\gamma} \log\log n.$$

Here γ is Euler's constant,

$$\gamma = \lim_{n\to\infty} \left(\left(\sum_{j=1}^{n} \frac{1}{j} \right) - \log(1+n) \right).$$

The nice surprise is that this inequality, with an explicit lower bound for n replacing "sufficiently large", and other related arithmetic inequalities and equalities, is equivalent to the Riemann hypothesis. The original inequality involving $\sigma(n)$ is the most famous, and the equivalence is known as "Robin's theorem". A detailed history of its evolution is given in [128] with quotes from Erdős, Hardy, Rankin, Berndt and Nicolas.

1.3 Volume One Summary

Except for Chapter 10, the text contains a mostly linear progression of ideas. In Chapter 2, key properties of the Riemann zeta function, which will be needed, are outlined and two key parameters developed. This is then followed in Chapter 3 by explicit estimates for functions of primes, given in such a way that their derivations can potentially be improved. Then in Chapter 4 some classical equivalences of the Riemann hypothesis are proved in full. These are used in the work that follows. In Chapter 5 a set of equivalences, for the most part in the form of inequalities, involving the Euler totient function $\varphi(n)$, are derived. Chapter 6 provides preparatory material for Chapter 7 by developing the properties of two types of so-called "abundant" numbers, wherein it is the number of divisors that are abundant. These are the numbers that appear as possible counterexamples to inequalities. In Chapter 7 an inequality based on values of $\sigma(n)$, the sum-of-divisors function, is shown to be equivalent to the hypothesis. In Chapter 8 the focus shifts to numbers not satisfying the inequality, with several results showing the numbers are very constrained, so sit in a narrow class. It is expected that this class will be further constrained. Chapter 7 already contained an alternative inequality, equivalent to the Riemann hypothesis, and Chapter 9 continues in this mode, expressing equivalent formulations for the hypothesis in terms of so-called "extraordinary" numbers and "extremely abundant" numbers.

The final chapter (Chapter 10) breaks the sequence of ideas, by giving ten other equivalent statements for the Riemann hypothesis, mostly proved in full. In one form or another, inequalities play a role in these formulations. The idea of this chapter is to reveal the ubiquitous nature of the hypothesis, and be the source of new ideas.

Further summary details are given in this section in the paragraphs below.

This first chapter, in Section 1.2, gives a sketch of the genesis of the Riemann hypothesis. Interestingly, it was not posed as a problem, conjecture or even hypothesis by its principal author, Bernhard Riemann, but we know that the issues therein were first publicly discussed in 1859. This date also marks the origin of what we now call the Riemann zeta function as a function of a complex variable, $\zeta(s)$.

Chapter 2 summarizes some basic properties of the Riemann zeta function, and gives an intuitive idea of its behaviour. Appendix H "The gamma function" and Appendix I "Riemann zeta function" in Volume Two [32] give more background and proofs. The fundamental parameters H and R are introduced and used throughout the text. The symbol H represents a y-value up to which all zeros of $\zeta(s)$ with positive imaginary part have been demonstrated to have their real part equal to 1/2. Comments are made, when needed, on which value of H has been chosen in a particular circumstance. The value of H that could be used is expected to increase in time, and theorems and algorithms are presented so new values can be adopted easily.

The value of R describes a simple form for a zero-free region of $\zeta(s)$, namely for $\zeta(\beta+i\gamma)=0$ we have

$$\beta < 1 - \frac{1}{R\log\gamma}.$$

It is expected that the value of R will decrease in time. We give the derivation of Landau, and quote more recent improvements.

Chapter 3 is devoted to numerical estimates, especially of the arithmetical functions $\theta(x)$ and $\psi(x)$. Note that there are additional estimates of products of functions of primes in Section 5.2. Estimates in this chapter are primarily derived without using RH. The material depends not only on the work of Rosser and Schoenfeld in their separate and joint papers of 1941, 1962, 1963 and 1975, but also on von Mangold's theorems of 1905.

Chapter 4 gives derivations of some well-known explicit classical equivalences to RH. It may be skipped by anyone familiar with introductory material. The proof of an important theorem of Landau, used in many cases in the text where RH is assumed to be false, is included. The chapter concludes with overview material on zero-free regions and a summary of heights up to which the hypothesis has been shown to hold, with the corresponding numbers of critical zeros.

The work of Rosser and Schoenfeld is remarkable in that it contains sharp explicit results produced before computer-based methods became ubiquitous. In this chapter we take some advantage of the much greater processing speed now available to extend the ranges for which computer verification of inequalities is practicable, and the greatly improved height below which all the zeros of $\zeta(s)$, with positive imaginary part, have real part $\sigma = \frac{1}{2}$. Improving these estimates is the subject of current research, since they have a strong influence on the equivalences to RH.

Moving on to Chapter 5, Nicolas proved in 1983 that RH is true if and only if for every non-prime primorial q we have

$$\frac{q}{\varphi(q)\log\log(q)} > e^{\gamma}.$$

Here a primorial is the product of all primes up to a given prime, so the kth primorial, say q, is $q = N_k := p_1 \cdots p_k$, where p_i is the ith prime starting with $p_1 = 2$.

Nicolas improved this result in 2012, and found four statements equivalent to the Riemann hypothesis. Let $N_k = 2\cdot 3 \cdots p_k$ be the kth primorial and let

$$c(n) := \left(\frac{n}{\varphi(n)} - e^{\gamma}\log\log(n)\right)\sqrt{\log n}$$

and define

$$\beta := \sum_{\rho} \frac{1}{\rho(1-\rho)} = 2 + \gamma - \log \pi - 2\log 2 = 0.046191\ldots,$$

where ρ ranges through the non-trivial zeros of $\zeta(s)$ with increasing absolute value of the imaginary part. Then RH is equivalent to each of the following:

(1) $\limsup_{n\to\infty} c(n) = e^{\gamma}(2+\beta) = 3.644415\ldots$.
(2) For all $n \ge N_{120569} = 2\cdot 3 \cdots 1591\,883$ we have $c(n) < e^{\gamma}(2+\beta)$.
(3) For all $n \ge 2$, $c(n) \le c(N_{66}) = c(2\cdot 3\cdots 317) = 4.0628356921$.
(4) For all $k \ge 1$ we have $c(N_k) \ge c(N_1) = c(2) = 2.208589\ldots$.

From this one shows that the Riemann hypothesis is equivalent to the following inequality holding for all $n \ge N_{120569}$:

$$\frac{n}{\varphi(n)} < e^{\gamma} \log\log n + \frac{e^{\gamma}(2+\beta)}{\sqrt{\log n}}.$$

The chapter includes the derivation of useful estimates for treating the Euler phi function, namely some products and sums of functions of primes. These depend on the relationship between primes and zeta zeros, and form an essential basis, along with other estimates, for developing the equivalences.

In Chapter 6, fundamental results to do with two types of number with many divisors are developed. These are called *superabundant* and *colossally abundant*. The work includes that of Alaoglu, Erdős and Nicolas. These numbers appear as counterexamples to RH, so are used in the chapters that follow. We include a corrected proof of a fundamental theorem of Alaoglu and Erdős.

Chapter 7 is also a fundamental chapter. Grönwall proved in 1913 that

$$G(n) := \frac{\sigma(n)}{n \log\log(n)} \quad \Longrightarrow \quad \limsup_{n\to\infty} \frac{\sigma(n)}{n\log\log n} = e^{\gamma} = 1.78107\ldots.$$

Ramanujan showed, probably in 1915, that if RH is true then for n sufficiently large we must have $G(n) < e^{\gamma}$. Then Robin showed in 1983 that RH is true if and only if $n > 5040$ implies $G(n) < e^{\gamma}$. Also included is the equivalence of Lagarias, which is dependent on the result of Robin, namely if $H_n = 1 + 1/2 + \cdots + 1/n$ is the n harmonic number, then RH is equivalent to the inequality

$$\sigma(n) \le H_n + \exp(H_n)\log(H_n), \quad n \ge 1.$$

In Chapter 8, properties of numbers that do not satisfy Robin's inequality are explored. The work of Choie, Lichiardopol, Moree and Solé is developed and extended. In particular it is shown that any integer greater than 5040 not satisfying Robin's inequality must be even, fail to be squarefree and not squarefull. The smallest such number must be a so-called Hardy–Ramanujan number and superabundant. Then it is shown, successively, that