The finite element method (FEM) is indispensable in the modeling and simulation of various engineering and physical systems, including structural analysis, stress, strain, fluid mechanics, heat transfer, dynamics, eigenproblems, design optimization, sound propagation, electromagnetics, and coupled field problems.

Incorporating theory, development of the method, and the use of FEM in the commercial sector, this textbook integrates basic theory with real-life, design-oriented problems using Ansys, the most commonly used computational software in the field.

For students as well as practicing engineers and designers, each chapter is highly illustrated and presented in a step-by-step manner. Fundamental concepts are presented in detail with reference to easy to understand worked examples that clearly introduce the method before progressing to more advanced mathematical content.

Included are step-by-step solutions for project type problems using modeling software, special chapters for modeling and the use of Ansys and Workbench programs, and extensive sets of problems and projects round out each chapter.

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Finite Elements for Engineers with Ansys Applications

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## Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Preface</td>
<td>xiii</td>
</tr>
<tr>
<td><strong>1 Finite Element Concepts</strong></td>
<td>1</td>
</tr>
<tr>
<td>1.1 General Solution of Continuum Problems</td>
<td>1</td>
</tr>
<tr>
<td>1.2 What is the Finite Element Method?</td>
<td>5</td>
</tr>
<tr>
<td>1.3 Basic Concepts and Definitions</td>
<td>13</td>
</tr>
<tr>
<td>1.4 Element Types and Degrees of Freedom (DOFs)</td>
<td>17</td>
</tr>
<tr>
<td>1.4.1 Truss Elements</td>
<td>17</td>
</tr>
<tr>
<td>1.4.2 Beam Elements</td>
<td>18</td>
</tr>
<tr>
<td>1.4.3 Two-Dimensional Elements</td>
<td>18</td>
</tr>
<tr>
<td>1.4.4 Three-Dimensional Shell Elements</td>
<td>20</td>
</tr>
<tr>
<td>1.4.5 Three-Dimensional Solid Elements</td>
<td>20</td>
</tr>
<tr>
<td>1.5 General Procedures for Finite Element Analysis</td>
<td>21</td>
</tr>
<tr>
<td>1.5.1 Basic Procedures in Finite Element Analysis</td>
<td>22</td>
</tr>
<tr>
<td>1.5.2 Phases of Finite Element Analysis in Commercial Programs</td>
<td>24</td>
</tr>
<tr>
<td>1.6 Brief History of the Development of the FE Method</td>
<td>26</td>
</tr>
<tr>
<td>1.6.1 Ancient Roots of the FE Method</td>
<td>26</td>
</tr>
<tr>
<td>1.6.2 History of the Development of the FE Method</td>
<td>28</td>
</tr>
<tr>
<td>1.6.3 History of the Development of Computers and FE Software Programs</td>
<td>30</td>
</tr>
<tr>
<td>Early Software Development</td>
<td>30</td>
</tr>
<tr>
<td>The Law-of-the-Land (Moore’s)</td>
<td>31</td>
</tr>
<tr>
<td>The Pre-Personal-Computer Era</td>
<td>32</td>
</tr>
<tr>
<td>The Personal-Computer Era</td>
<td>32</td>
</tr>
<tr>
<td>64 Bits and Parallelization</td>
<td>34</td>
</tr>
<tr>
<td>1.6.4 Future Outlook and What’s Next?</td>
<td>34</td>
</tr>
</tbody>
</table>
# Contents

1.7 Finite Element Applications .................................................. 36
  1.7.1 Various Types of Application .................................................. 38
Problems .................................................................................. 43

2 Detailed Procedures ................................................................. 50
The intention of this chapter is to provide basic understanding of the FE concepts, underlying assumptions and simple element developments without indulging in deep mathematical treatments. Concepts of stiffness transformation and imposition of simple boundary conditions are discussed and examples are provided.

Chapter Roadmap .................................................................. 50
2.1 Element Characteristic Equations for Simple Elements .......... 50
  2.1.1 Simple Truss Element – One DOF per node ..................... 50
  2.1.2 One-Dimensional Heat Transfer Element ...................... 55
  2.1.3 Pipe Flow Element .......................................................... 57
  2.1.4 Direct Current Flow Element ........................................... 59
  2.1.5 Torsional Element .......................................................... 59
2.2 Simple Beam Element .......................................................... 61
2.3 Assembling of Global Equations ............................................. 72
2.4 Application of Boundary Conditions ..................................... 81
  2.4.1 Rigid Body Motion .......................................................... 82
  2.4.2 Applying Boundary Conditions ......................................... 82
  2.4.3 Application Example: Beam Element with a Hinge .......... 85
2.5 Coordinate Transformations and More Element Equations ..... 97
  2.5.1 Coordinate Transformation .............................................. 97
  2.5.2 General Two-Dimensional Truss Element ....................... 99
  2.5.3 General Two-Dimensional Beam Element ............... 100
  2.5.4 Three-Dimensional Truss Element ................................. 101
  2.5.5 Three-Dimensional Beam Element ............................... 103
Problems .................................................................................. 109

3 Modeling Aspects ................................................................. 130
This chapter provides practical guidelines for the modeling phase in finite element analysis. The chapter highlights geometric modeling, problem classification, element choice, symmetry and boundary conditions, mesh intensity and lumped loading. Some discussion of model accuracy and mesh refinement is given.

Chapter Roadmap ................................................................ 130
3.1 The Modeling Process ......................................................... 130
3.2 Geometric Modeling .......................................................... 131
  Key Points ............................................................................ 133
4 Linear Static Analysis Using Ansys/Workbench

This chapter starts by giving a brief introduction to the use of the Ansys program. Then, the chapter provides a step-by-step example for using Ansys in the analysis of two-dimensional linear elastic
problems. The second part of the chapter provides an introduction to
the Workbench program and this is followed by two step-by-step
eamples of using Workbench in solving two- and three-
dimensional linear elastic problems.

Chapter Roadmap

4.1 Introduction to the Ansys Program 198
4.1.1 Operation Modes in Ansys 198
4.1.2 Starting up Ansys 199
4.1.3 Windows in Ansys 200
4.1.4 Ansys Structure and Files 201

4.2 Ansys Project 1: Analysis of a 2D Support Bracket 202
4.2.1 Problem Description 202
4.2.2 Create the Model Geometry 203
4.2.3 Generate the FE Model 211
4.2.4 Solution Operation 215
4.2.5 Displaying the Results and Exiting Ansys 217
4.2.6 Batch Type Solution 219
4.2.7 Mesh Sensitivity Analysis 221

4.3 Introduction to Workbench 225
4.3.1 Layout of Program Menus 225
4.3.2 Overview of the Geometry and the DesignModeler
Capabilities 227

4.4 Workbench Project 1: 2D Cantilever Bracket 232
4.4.1 Create the Model Geometry 232
4.4.2 Generate the FE Model 238
4.4.3 Solution, Results and Mesh Refinement 240

4.5 Workbench Project 2: 3D T-Junction Analysis 244
4.5.1 Problem Description 245
4.5.2 Creating the Geometry 246
4.5.3 T-Junction FE Model and Solution 254

5 Finite Element Formulations 266
This chapter examines two basic methods of formulation in
finite elements; the virtual work principle and the weighted
residual Galerkin method. Examples of the use of each method in
linear static, heat transfer and field problems are provided.

Chapter Roadmap

5.1 Overview 266
5.2 Strong and Weak Forms of the Problem 267
5.3 FE Virtual Work Formulations 271
5.4 Principle of Minimum Potential Energy and the Rayleigh–Ritz Method 276
5.5 Linear Elastic FE Analysis Using Virtual Work Principle 280
5.6 Heat Conduction Analysis Using Virtual Work Principle 283
5.7 Method of Weighted Residuals (MWRs) 286
  5.7.1 Overview of MWRs 286
  5.7.2 Various Weighted Residual Methods 288
    Point and Subdomain Collocation Methods 288
    Continuous Least Squares Method 289
    Least Squares Collocation Method 290
    Galerkin Methods 291
5.8 Galerkin Formulation of 2D Linear Elasticity Problems 302
5.9 Formulation of Heat Conduction Analysis 313
  5.9.1 Heat Conduction Analysis Using Variational Principles 313
  5.9.2 3D Heat Conduction Analysis Using the Galerkin Formulation 317
Problems 322
Project Type Problems 327

6 Linear Static Analysis 335
This chapter highlights linear elastic analysis formulation; application to various elements; displacement based elements; 2D plane stress, plane strain and axisymmetric elements; 3D elements; thermal stress analysis; isoparametric elements; and application of linear constraint equations.
Chapter Roadmap 335
6.1 Linear Elastic Analysis Formulation 335
  6.1.1 Problem Formulation by the Virtual Work and Galerkin Principles 336
  6.1.2 Specialization to Two-Dimensional Problems 338
6.2 Application to General Three-Dimensional Solid Elements 341
6.3 Development of Simple Beam Element 344
6.4 Development of Two-Dimensional Elements 353
  6.4.1 Plane Stress or Strain Triangular Element with Three Nodes 354
  6.4.2 Plane Stress or Strain Rectangular Element with Four Nodes 366
  6.4.3 Axisymmetric Triangular Ring Element with Three Nodes 372
6.5 Development of Three-Dimensional Solid Elements 378
  6.5.1 Tetrahedron Element with Four Nodes 378
## Contents

6.6 Thermal Stresses 384
  6.6.1 Constitutive Equations with Thermal Effect 384
  6.6.2 Element Equations with Thermal Effect 388

6.7 Isoparametric Elements 397
  6.7.1 General Concepts and the Need for Isoparametric Elements 397
  6.7.2 Development of 1D Truss Element 399
  6.7.3 Development of 2D Quadrilateral Element with Four to Nine Nodes 401
  6.7.4 Development of 2D Triangular Element with Three to Six Nodes 413
  6.7.5 Development of 3D Hexahedral or Cubical Element with Eight to 20 Nodes 420
  6.7.6 Development of 3D Tetrahedral Elements 424

6.8 Application of Multi-Point Linear Constraint Equations (MPCs) 428
  6.8.1 Types of Constraint 428
  6.8.2 Method of Substitution 429
  6.8.3 Lagrange Multiplier Method 430
  6.8.4 Penalty Method 433

Problems 440

Multiple Choice Questions 468

7 Conduction Heat Transfer Analysis 476

This chapter starts by developing the three-dimensional transient conduction heat transfer equations and identifying various boundary conditions and sources of nonlinearity. The chapter then discusses methods of solution, time discretization schemes and the accuracy and stability of various schemes. Various elements for one-, two- and three-dimensional problems are developed and detailed examples are presented.

Chapter Roadmap 476

7.1 General Background 476
  7.1.1 Basic Modes of Heat Transfer 476
    Conduction 476
    Convection 478
    Radiation 478
  7.1.2 The One-Dimensional Heat Conduction Equation 479
  7.1.3 Boundary Conditions 481
  7.1.4 Sources of Nonlinearity 483

7.2 FE Formulation of the Heat Conduction Problem 484
## Contents

7.3 Solution of the Transient Heat Transfer Equations 492
   7.3.1 General Considerations 492
   7.3.2 Direction Integration Method 493
      Linearization of the General Equation 493
      The $\alpha$-Method 495
      General Solution Steps 497
      Stability and Accuracy Considerations 498
      Accuracy of the $\alpha$-Method 499

7.4 Element Equations for Heat Transfer Analysis 501
   7.4.1 One-Dimensional Elements 501
   7.4.2 Two-Dimensional Elements 502
      Three-Node Plane Triangular Elements 502
      Three-Node Axisymmetric Elements 503
      Isoparametric Elements 505

7.5 Ansys/Workbench Project: Square Plate with Prescribed Temperature Boundary Conditions 518
   Problem Statement 518
   Approach and Assumptions 518
   Solution Steps 518

Problems 526

Project Type Problems 537

Appendix A Matrices and Systems of Linear Equations 543
Appendix B Vectors and Tensors 581
References 603
Index 610
The finite element method (FEM) is broadly defined as a numerical approach for approximating the governing equations of any continuous system. The method has evolved into an indispensable technology in the modeling and simulation of various engineering and physical systems. The advent and advancement of digital computers have enabled the application of the method in all fields of engineering. The FEM has a number of distinctive features and advantages which make it superior to other numerical approaches. It is adaptable for solving problems with general geometry and boundary conditions and may easily incorporate nonlinear effects, different material models, thermal effects and coupled phenomena. Essentially, it may be applied to any physical problem described by the equations of calculus, e.g. differential, integral, or variational equations. Additionally, FEM computer programs have become common tools in the hands of engineers, designers and analysts.

This text is intended to target users of the FEM for design and analysis from both academic and industrial environments. There is a large number of finite element texts in the market. On one hand, many of the available texts rigorously discuss the theoretical aspects and developments of the method and on the other hand some texts concentrate mainly on the practical use of the method and commercial programs. This text strives to strike a balance between the two approaches so that it will be appealing to senior undergraduate students and beginner graduate students as well as practicing engineers and designers. Before indulging in mathematical treatments, the text highlights the basic concepts of the FEM with minimal mathematical involvement. From the decades of teaching experience of the author, this has proved to help students understand the theory and become confident in using programs. The philosophy is to let the reader master the basic concepts before digging into rigorous mathematical development. The conceptual and theoretical chapters in the text are full of simple hand-calculation and Matlab problems. Each of these chapters concludes with a detailed project that provides step-by-step examples using one of the major FE commercial programs (Ansys or Workbench). However, the first three chapters are followed by a special application chapter with a brief introduction to Ansys and Workbench as well as some project-type problems. Hence, this book integrates the basic theory with real-life, design-oriented problems which involve practicing and using some of the most common computational software program in the field. Material is presented in a step-by-step manner in each chapter and fundamental concepts are presented in
detail with reference to simple and easy-to-understand problems, assuming that
the reader has no previous knowledge of the method. This approach improves and
enhances readability, reduces frustration and increases retention. At the end of
each section, there is synopsis box outlining the main concepts, important equa-
tions and conclusions of the section. At the end of each chapter, there is a large
selection of hand-calculation, Matlab, Ansys/Workbench and design problems
that have been collected over decades of teaching. Most of the problems are solved
in a supplementary solution manual.

The book is organized into seven chapters. The first four chapters are devoted to
discussions of various FEM concepts along with the practical use of the Ansys and
Workbench programs. Chapter 1 outlines the general steps of finite element anal-
ysis (FEA) through simple geometric problems, provides conceptual definitions to
various terms used in the analysis and gives a brief history of the method and its
various applications. The basic concepts, the phases of using the method, the under-
lying assumptions, simple element stiffness developments and the transformation
and imposition of simple boundary conditions are discussed in Chapter 2 with-
out going into deep mathematical treatments. Chapter 3 provides practical guidelines
for the modeling phase in FEA. This chapter highlights geometric modeling, problem classification, element choice, symmetry and boundary conditions, mesh
intensity and lumped loading. Chapter 4 gives a brief introduction to the use of
the Ansys and Workbench programs along with step-by-step examples for using
them in the analysis of two- and three-dimensional linear elastic problems. This
chapter may be considered as an application of the concepts in Chapter 3 and,
as a logical consequence, the problem sets for these two chapters are given after
Chapter 4. Chapter 5 marks the start of the rigorous mathematical treatment of the
finite element method. This chapter examines two basic methods of formulation
in finite elements: the virtual work principle and the weighted residual Galerkin
method. Examples of the use of each method in linear static, heat transfer and
field problems are provided. Chapter 6 introduces the first rigorous applications
to linear static analysis problems. This chapter starts by highlighting the general
formulation for linear static analysis problems and then details various element
applications for beam, plane stress/strain, axisymmetric and three-dimensional
solid elements. Following the element development, a section on thermal stress
analysis is provided, which is followed by a detailed discussion of isoparametric
elements. A final section on handling linear multi-point constraint equations is
provided; this is followed by an extensive problem set, with projects. The final
chapter discusses conduction heat transfer analysis, and it starts with a brief intro-
duction to three-dimensional transient conduction analysis in solids with con-
vection and radiation boundary conditions. Then, the chapter discusses various
methods for solving the transient equation and provides detailed matrix equations
for the analysis along with sample element developments. The chapter ends with
detailed analysis problems and an Ansys/WB project. The treatment in this chapter is rather condensed and brief; it is meant to provide a quick review of the basic equations in the field of application with minimal mathematical treatment. Finally, some of the necessary background is provided in Appendices A and B. These include: Matrices and Systems of Linear Equations (Appendix A), and Vectors and Tensors (Appendix B).

Gratitude is owed to many friends, colleagues and researchers who have contributed through discussions, ideas and suggestions to improve the text and bring it to its current format. My journey with the field has spanned 45 years, starting at Cairo University where I was introduced to the principles of mechanics by my uncle Abdul-Aziz Gadala and to FEA by the late Professor Salah Bayoumi. The final part of my journey was at the University of British Columbia (UBC) in Vancouver, where I spent over 25 years teaching design and FEA. Between the two stops, I passed by many stations including McMaster University, Atomic Energy of Canada, Ontario Hydro Research, Engineering Mechanics Research Corporation in Michigan and the University of Michigan. During this journey, I have collaborated with numerous industrial partners in Canada, Europe, the United States and the Middle East. Undoubtedly, this collaboration has contributed significantly to this work and elevated it to the current format. Without such collaboration and the fruitful ideas and help of my own students at UBC this book could not have been written.

Finally, special thanks to my wife Omnia and my children, Drs. Marwa, Mariam, and Ibrahim, for their immense support and forbearance with the numerous occasions when the task of writing has deprived them of invaluable family time together.