1 Introduction

In this chapter, we first provide some motivation for the type of modeling problems we address in this book. Then we provide an overview of the types of mathematical model used to describe the behavior of the classes of systems of interest. We also describe the types of uncertainty model adopted and how they fit into the mathematical models that describe system behavior. In addition, we provide a preview of the applications discussed throughout the book, mostly centered around electric power systems. We conclude the chapter by providing a brief summary of the content of subsequent chapters.

1.1 Motivation

Loosely speaking, an engineered system is a collection of hardware and software components assembled and interacting in a particular way so that they collectively fulfill some function. The interaction between components can be of physical nature, i.e., components can be electrically, mechanically, or thermally coupled, and thus may involve some exchange of energy. Components can also be coupled in the sense that they exchange information with each other. Because of phenomena external to the system, there is some uncertainty as to how the system will perform. This phenomenon can materialize as an external (time-varying) input that drives the system response, or as a change in the system structure. In both cases, these external phenomena will alter the system nominal response and might cause the system to fail to perform its function. While most systems are typically designed to withstand some structural and operational uncertainty, it is important to verify that this is the case before the system is deployed.

To illustrate these ideas, consider the power supply system in Fig. 1.1, whose function is to reliably provide electric power to a mission-critical computer load, labeled as IT load, at a certain voltage level. To this end, there are three sources of power: two utility feeders, labeled as feeder 1 and feeder 2, and a backup generator, labeled as genset. Having such a redundant arrangement ensures delivery of power to the IT load with high assurance. While not depicted in the figure, there is a computer-based control system in charge of monitoring and controlling the power sources and switchgear, which plays an important role in the analysis of the system.

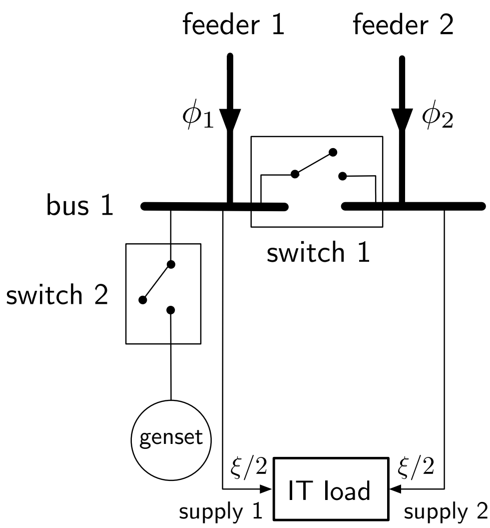

Figure 1.1 Schematic of power supply system for mission-critical load, where ϕ_1 and ϕ_2 denote the active power flowing through feeder 1 and feeder 2, respectively, and ξ denotes the total power delivered to the IT load.

Under normal operating conditions, both switch 1 and switch 2 are open (therefore genset is not initially used to supply power to the IT load). Then, feeder 1 supplies power to bus 1, which, in turn, serves half of the power demanded by the IT load via supply 1. Similarly, feeder 2 supplies bus 2, which in turn serves the other half of the power demanded by the IT load via supply 2. The total power demanded by the IT load is determined by the computer workload, which can vary according to external requests received by the computer and is a priori unknown. Thus, from the point of view of the power supply system, the computer workload is an external input that drives the power supply system response. Since the workload evolves over time and is unknown a priori, there is some uncertainty on how the power supply system will perform its function.

In the event that there is an outage in either feeder 1 or feeder 2, switch 1 will close and all power to the IT load will be supplied by the available feeder. In the event that there is an outage in both feeders (either sequential in time or simultaneous), switch 2 is closed and the genset will supply all the power to the IT load until either (i) one or both utility feeders are restored back to operation, at which point in time, switch 2 is open and the power to the IT load is once again supplied by one or both utility feeders, or (ii) there is an event that causes the genset to fail, at which point the IT load is shut down offline. Thus, the phenomena causing feeder outages and genset failure result in a change in the system structure in the sense of how power is routed from the available sources to the IT load.

In the context of the system above, one might be interested in quantifying the impact of workload variability on certain variables of interest, e.g., the flows of power through the wires connecting the buses and the IT load, and the magnitude of the voltage at bus 1 and bus 2 (when not connected together). This analysis is necessary to ensure that wires are sized correctly and protection equipment, e.g., under- and over-voltage protection relays, is calibrated appropriately. It is also necessary to ensure that after outages in one or both feeders occur, subsequent switching actions are correctly executed. In addition, one could be interested in quantifying the impact of equipment failure on the system ability to perform its function over some period of time.

The goal of this book is to develop analysis tools to perform the types of analysis described above. The applications and examples throughout the book draw heavily from electric power applications, including bulk power systems and microgrids, and linear and switched linear circuits encountered in power electronics applications. However, the modeling framework and techniques presented are general and can be applied to other engineering domains, including automotive and aerospace applications. For example, they can be used to assess the dynamic performance of an automotive steer-by-wire system and the lateral-directional control system of a fighter aircraft.

1.2 System Models

In a broad sense, one can think of an engineered system as an entity imposing constraints on certain *variables* associated with the system energy and information content. With this point of view, we can represent the behavior of the system by a set of mathematical relations between the aforementioned variables; this is what we refer to as the model of the system. These relations can be a result of physical laws, e.g., Kirchhoff's laws, energy conservation law, or moment conservation law. They can also arise from algorithms implemented in a digital computer, for modifying (controlling) the physical behavior of the system, e.g., the proportional-integral control scheme used in a bulk power system to automatically regulate frequency across the system. These mathematical relations will, in general, also include numerous *parameters*, i.e., quantities defining physical or information properties of the components comprising the system. Such system parameters can be constant or vary with time, and their value can be a priori unknown or uncertain. When modeling a system, the distinction between parameters and variables is typically clear because the values taken by the system parameters should not be affected by the values taken by the system variables or other parameters. Indeed, if the value of some parameter p is actually affected by the values taken by the system variables or other parameters, then the model should reflect this dependence and instead of being treated as a parameter in the model, p should be considered as an additional system variable and treated in the model as such. Next, we illustrate the ideas above via some examples.

Example 1.1 (Power supply system) Consider the system in Fig. 1.1 and assume there are no losses in any of its components. Let ϕ_1 and ϕ_2 denote the active power flowing through feeder 1 into bus 1 and the active power flowing through feeder 2 into bus 2, respectively. Let p_g denote the active power supplied by the genset. Let ξ denote the active power demanded by the IT load. Recall that under normal operating conditions, half of the power to the IT load is supplied by feeder 1, while the other half is supplied by feeder 2 (the genset does not supply any power). Then, since the active power flowing in and out of both bus 1 and bus 2 needs to be balanced, we have that

$$\phi_1 = \frac{\xi}{2}, \quad \phi_2 = \frac{\xi}{2}, \quad p_g = 0. \tag{1.1}$$

Now, recall that if there is an outage in feeder 1 (feeder 2), switch 1 will close and all the power to the IT load will be delivered by feeder 2 (feeder 1); thus,

$$\begin{aligned} &\phi_1 = 0, \quad \phi_2 = \xi, \quad p_g = 0, \quad \text{if outage in feeder 1 and feeder 2 in service,} \\ &\phi_1 = \xi, \quad \phi_2 = 0, \quad p_g = 0, \quad \text{if outage in feeder 2 and feeder 1 in service.} \end{aligned} \tag{1.2}$$

Also, recall that if there is an outage in both feeders, the genset will supply all the power to the load; thus,

$$\phi_1 = 0, \quad \phi_2 = 0, \quad p_g = \xi. \tag{1.3}$$

Finally, for the case when there is an outage in both feeders and a failure in the genset, we have that

$$\phi_1 = 0, \quad \phi_2 = 0, \quad p_g = 0, \tag{1.4}$$

and the IT load is shut down offline; thus, $\xi = 0$. Note that the relations in (1.1–1.4) only involve ξ, ϕ_1, ϕ_2, and p_g, which in this case are the system variables, i.e., the model here does not involve any parameters.

Example 1.2 (Linear circuit) Consider the circuit in Fig. 1.2 and assume that $c(t) = c,\ t \geq 0$, where c is a positive scalar. First, note that

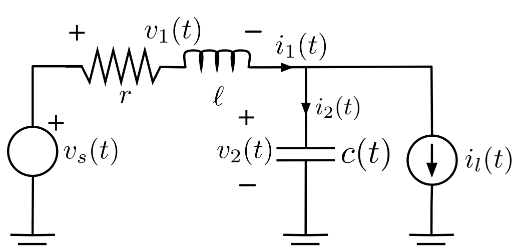

Figure 1.2 Linear circuit.

$$v_1(t) = r\, i_1(t) + \ell \frac{di_1(t)}{dt},$$
$$i_2(t) = c \frac{dv_2(t)}{dt}. \tag{1.5}$$

Then, by using Kirchhoff's laws, we can obtain the following model describing the relation between the currents $i_1(t)$ and $i_l(t)$, and the voltages $v_s(t)$ and $v_2(t)$:

$$0 = i_1(t) - c\frac{dv_2(t)}{dt} - i_l(t),$$
$$0 = v_s(t) - r\, i_1(t) - \ell \frac{di_1(t)}{dt} - v_2(t), \tag{1.6}$$

where r, ℓ, and c are positive constants. Here, $i_1(t)$, $i_2(t)$, $v_s(t)$, and $v_2(t)$ are variables, whereas r, ℓ, and c are parameters. Now, assume that $c(t)$ is known to evolve according to

$$\frac{dc(t)}{dt} = -c(t) + \alpha u(t),$$

where α is a positive scalar; thus,

$$i_2(t) = c(t)\frac{dv_2(t)}{dt} + v_2(t)\big(-c(t) + \alpha u(t)\big).$$

Then, the model describing the circuit behavior is as follows:

$$0 = i_1(t) - c(t)\frac{dv_2(t)}{dt} - v_2(t)\big(-c(t) + \alpha u(t)\big) - i_l(t),$$
$$0 = v_s(t) - r i_1(t) - \ell \frac{di_1(t)}{dt} - v_2(t),$$
$$0 = \frac{dc(t)}{dt} + c(t) - \alpha u(t); \tag{1.7}$$

thus, in this model, the capacitance, $c(t)$, is no longer a parameter but a variable.

There are some fundamental differences between the models in (1.1–1.4), (1.6), and (1.7). First, in the models in (1.1–1.4) and (1.6), the relation between the variables is linear, whereas in the model in (1.7), the relation between the variables is nonlinear. Second, in the model in (1.1–1.4), the constraints imposed on the system variables are in the form of a system of algebraic equations, whereas in the models in (1.6) and (1.7), the constraints imposed on the system variables are in the form of a set of ordinary differential equations (ODEs). In this book, we refer to systems whose behavior can be described by a set of algebraic equations as *static systems*, whereas systems whose behavior can be described by a set of ODEs are referred to as *continuous-time dynamical systems*. Furthermore, we refer to systems whose behavior can be described iteratively by a set of recurrent relations as *discrete-time dynamical systems*.

So far, we have not discussed the nature of the variables describing the energy and information state of a system; in general, we will categorize them as either inputs or states. By inputs, we refer to variables that are set and can be varied extraneously, whereas by states, we refer to variables that result from the constraints describing the system behavior and the values the inputs take. With this categorization, we can rewrite the static system in (1.1–1.4) as follows:

$$x = H_i\xi, \quad i = 1, 2, 3, \tag{1.8}$$

where $x = [\phi_1, \phi_2, p_g]^\top$, $\xi \geq 0$, and

$$H_1 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix}, \quad H_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad H_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \tag{1.9}$$

and

$$x = H_4\xi, \tag{1.10}$$

where $\xi = 0$, and

$$H_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{1.11}$$

More generally, in this book we will consider static systems of the form

$$x = h_i(w), \quad i \in \mathcal{Q}, \tag{1.12}$$

where $x \in \mathbb{R}^n$ is referred to as the state vector, $w \in \mathbb{R}^m$ is referred to as the input vector, $\mathcal{Q}$ takes values in some finite set, and $h_i \colon \mathbb{R}^m \to \mathbb{R}^n$, which we refer to as the system input-to-state mapping, is defined by the relations between the state variables (i.e., the entries of the state vector), the inputs (i.e., the entries of the input vector), and the system parameters.

By using the same categorization of variables as inputs or states, we can rewrite the model in (1.6) in state-space form as follows:

$$\frac{d}{dt}x(t) = Ax(t) + Bw(t), \quad t \geq 0, \tag{1.13}$$

where $x(t) = \big[v_2(t), i_1(t)\big]^\top$ is referred to as the state vector, $w(t) = \big[v_s(t), i_l(t)\big]^\top$ is referred to as the input vector, and

$$A = \begin{bmatrix} 0 & \frac{1}{c} \\ -\frac{1}{\ell} & -\frac{r}{\ell} \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -\frac{1}{c} \\ \frac{1}{\ell} & 0 \end{bmatrix}. \tag{1.14}$$

More generally, we will consider continuous-time dynamical systems of the form

$$\frac{d}{dt}x(t) = f\big(t, x(t), w(t)\big), \quad t \geq 0, \tag{1.15}$$

where $f \colon [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, is defined by the relations between the state variables (i.e., the entries of the state vector, $x \in \mathbb{R}^n$); and the inputs (i.e., the entries of the input vector, $w(t) \in \mathbb{R}^m$). For example, the system in (1.7) can be written as

$$\frac{d}{dt}x(t) = f\big(x(t), w(t)\big), \tag{1.16}$$

where $x(t) = \big[v_2(t), i_1(t), c(t)\big]^\top$, $w(t) = \big[v_s(t), i_l(t), u(t)\big]^\top$, and

$$f\big(x(t), w(t)\big) = \begin{bmatrix} \frac{1}{c(t)}\big(v_2(t)c(t) - \alpha v_2(t)u(t) + i_1(t) - i_l(t)\big) \\ \frac{1}{\ell}\big(- v_2(t) - r\, i_1(t) + v_s(t)\big) \\ -c(t) + \alpha u(t) \end{bmatrix}. \tag{1.17}$$

Finally, in this book we also consider discrete-time dynamical systems of the form

$$x_{k+1} = h_k(x_k, w_k), \quad k = 0, 1, 2, \ldots, \tag{1.18}$$

where $x_k \in \mathbb{R}^n$, $w_k \in \mathbb{R}^m$, and $h_k \colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $k = 0, 1, 2, \ldots$, is defined by the relations between the state variables and inputs of the particular system under consideration.

1.3 Uncertainty Models

Here, we provide an overview of the different uncertainty models considered in subsequent chapters for both static and dynamical systems. In the process, we also state the analysis objectives for both classes of systems under each uncertainty model considered.

1.3.1 Static Systems

Consider a static system of the form:

$$x = h_i(w), \quad i \in \mathcal{Q}. \tag{1.19}$$

If the value that w takes is uncertain, we say the system is subject to input uncertainty, whereas if the value that i takes is uncertain, we say the system is subject to structural uncertainty. In terms of formally describing input uncertainty, we will consider both probabilistic models and set-theoretic models. In terms of formally describing structural uncertainty, we will only consider a probabilistic model. Each of these models is briefly described next.

Probabilistic Input Uncertainty Model: We assume the values that w can take are described by a random vector with known first and second moments or known probability density function (pdf). Then, for each $h_i(\cdot)$, $i \in \mathcal{Q}$, the values that $x = h_i(w)$ can take will be also random and described by some random vector, and the objective is to characterize the first and second moments (or the pdf) of this random vector.

Set-Theoretic Input Uncertainty Model: We assume the values that the input vector, w, can take belong to some convex set. While in general this set can have any shape, in this book we restrict our analysis to two particular classes of closed, convex sets, namely ellipsoids and zonotopes. Then, for each $h_i(\cdot),\ i \in \mathcal{Q}$, the possible values that the state, $x = h_i(w)$, can take will belong to some set, and the objective is to characterize such a set.

Structural Uncertainty Model: We assume that the input-to-state mapping, $h_i(\cdot)$, evolves with time according to a Markov chain with known transition probabilities. Thus, for a given w, the value that $x = h_i(w)$ takes will evolve according to the aforementioned Markov process. Here, the value that w can take is either a known constant or can be described by a probabilistic model like the one above. Then, the objective is to characterize the state vector statistics.

1.3.2 Dynamical Systems

Consider continuous-time systems of the form

$$\frac{d}{dt}x(t) = f\big(t, x(t), w(t)\big), \tag{1.20}$$

and discrete-time systems of the form

$$x_{k+1} = h_k(x_k, w_k). \tag{1.21}$$

Here we will only analyze the system behavior under input uncertainty, i.e., the values that $w(t)$ and w_k can take are not a priori known, and will use both probabilistic and set-theoretic models to describe them.

Probabilistic Input Uncertainty Model: For continuous-time dynamical systems, we consider the case when the function $f(\cdot,\cdot,\cdot)$ is defined as follows:

$$f(t, x, w) = \alpha\big(t, x) + \beta(t, x)w, \tag{1.22}$$

whereas, for discrete-time dynamical systems, we do not impose any restrictions on $h_k(\cdot,\cdot)$. We assume that the values that $w(t)$ and w_k take are random and governed by a "white noise" process; thus, the values that $x(t)$ and x_k take are also random and governed by some stochastic process. We first consider the case when we only know the mean and covariance functions of the "white noise" input process and characterize the mean and covariance functions of the stochastic process describing the evolution of $x(t)$ and x_k. Then, we further assume that the "white noise" input process is Gaussian and provide the complete probabilistic characterization of the stochastic process describing the evolution of $x(t)$ and x_k.

Set-Theoretic Input Uncertainty Model: We consider general functions $f(\cdot,\cdot,\cdot)$ and $h_k(\cdot,\cdot)$, and assume the values that $w(t)$ and w_k can take are known to belong to a convex set, namely an ellipsoid. Then, the values that $x(t)$ and x_k can take also belong to some set (not necessarily an ellipsoid), and the objective is to characterize such a set. Providing an exact characterization of such a set is difficult in general (even if $f(\cdot,\cdot,\cdot)$ and $h_k(\cdot,\cdot)$ are affine functions); thus, we settle for obtaining ellipsoidal upper bounds.

1.4 Application Examples

Most of the techniques presented in the book are illustrated by using examples from circuit theory, electric power systems, and power electronics. For example, we utilize a simplified formulation of the power flow problem in AC power systems to illustrate the techniques developed for analyzing static systems subject to input uncertainty. Also, in order to illustrate the analysis techniques for dynamical systems subject to input uncertainty, we utilize a simplified model of the dynamics of an inertia-less AC microgrid, i.e., a small AC power system whose generators and loads are interfaced with the network via power electronics.

1.4.1 Power Flow Analysis under Active Power Injection Uncertainty

Consider a three-phase power system comprising n buses $(n > 1)$ indexed by the elements in $\mathcal{V} = \{1, 2, \dots, n\}$, and l transmission lines $\big(n-1 \leq l \leq n(n-1)/2\big)$ indexed by the elements in the set $\mathcal{L} = \{1, 2, \dots, l\}$, and assume the following hold:

A1. The system is balanced and operating in sinusoidal regime.
A2. There is at most one transmission line connecting each pair of buses.
A3. Each transmission line is short and lossless.
A4. The voltage magnitude at each bus is fixed by some control mechanism.

Let p_i denote the active power injected into the system network via bus i, and let ϕ_e denote the active power flowing on transmission line e, $e = 1, 2, \dots, l$. Assume that

$$p_i = \xi_i, \quad i = 1, 2, \dots, n-1,$$

where ξ_i is extraneously set and a priori unknown. Then, since the system is lossless, the injection into bus n must be such that $\sum_{j=1}^{n} p_j = 0$; thus,

$$p_n = -\sum_{j=1}^{n-1} \xi_j.$$

Define $\xi = [\xi_1, \xi_2, \dots, \xi_{n-1}]^\top$ and $\phi = [\phi_1, \phi_2, \dots, \phi_l]^\top$. Then, by imposing some conditions on the values that ξ can take, there exists a function $f\colon \mathbb{R}^l \to \mathbb{R}^l$ encapsulating the system network topology and transmission line parameters such that

$$w = f(\phi), \tag{1.23}$$

where $w = \left[\xi^\top, \mathbf{0}_{l-n+1}^\top\right]^\top$. The formulation above can be generalized to the case when there are m, $1 \leq m \leq n-1$, power injections being extraneously set with the remaining power injections being adjusted so that $\sum_{j=1}^{n} p_j = 0$, which is a necessary condition that the power injections need to satisfy because of the assumption on the transmission lines in the system being lossless.

If the network is a tree, then we have that $l = n - 1$ and

$$\xi = \widetilde{M}\phi,$$

where $\widetilde{M} \in \mathbb{R}^{(n-1)\times(n-1)}$ is invertible; thus, we can write

$$\phi = \widetilde{M}^{-1}\xi. \tag{1.24}$$

For the case when the network is not a tree, because of the assumptions made on the values that ξ can take, we can ensure that there exists $f^{-1}: \mathbb{R}^l \to \mathbb{R}^l$ such that

$$\phi = f^{-1}(w). \tag{1.25}$$

Then, given either a probabilistic model or a set-theoretic model describing the values that the vector of extraneous power injections, ξ, can take, the problem is to characterize the values that the vector of line flows, ϕ, can take. We explore such settings in detail in Chapter 3 and Chapter 7.

1.4.2 Analysis of Inertia-less AC Microgrids under Power Injection Uncertainty

Consider a three-phase microgrid comprising n buses $(n > 1)$ indexed by the elements in $\mathcal{V} = \{1, 2, \ldots, n\}$, and l transmission lines $\big(n - 1 \leq l \leq n(n-1)/2\big)$ and assume the following hold:

B1. The microgrid is balanced and operating in sinusoidal regime.
B2. There is at most one transmission line connecting each pair of buses.
B3. Each transmission line is short and lossless.
B4. Connected to each bus there is either a generating- or a load-type resource interfaced via a three-phase voltage source inverter.
B5. The reactance of each voltage source inverter output filter is small when compared to the reactance values of the network transmission lines.
B6. The phase angle of the inverter connected to each bus is regulated via a droop control scheme updated at discrete time instants indexed by $k = 0, 1, 2, \ldots$.
B7. The inverter outer voltage and inner current control loops hold the inverter output voltage magnitude constant throughout time.

Consider the case when the frequency-droop control setpoints of the inverters at buses $1, 2, \ldots, m$ change randomly (this is typically the case in photovoltaic installations endowed with maximum power point tracking). Let ξ_k denote an m-dimensional vector whose entries correspond to the values taken at instant k of said setpoints. Assume that in order to mitigate the effect of these random fluctuations, the frequency-droop control setpoints of the inverters at buses $m+1, m+2, \ldots, n$ are regulated via a closed-loop integral control scheme so that the frequency across the microgrid network remains close to some nominal value. Let