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Infinite-Dimensional Models in Statistical Physics

Infinite-dimensional models are core to statistical physics. They can be used to understand liquids and glasses, as they are in this book [153, 206, 292, 362], but also strongly coupled electrons [164], atomic physics [337] and gauge field theory [132], to name a few. The reason is that infinite-dimensional models are exactly solvable using mean field methods. The aim of this chapter is to give an example of this construction in the context of the Ising model of magnetism.

It will be assumed that the reader is already familiar with the basic properties of the Ising model as presented, for example, in the first chapters of [69]: its main observables (magnetisation, magnetic susceptibility), its phase diagram and phase transitions, and its dynamics. The aim of this chapter is mostly to present these properties in the context of a large dimensional expansion and to introduce the concept of a thermodynamic (stable or metastable) state, identified with a local minimum of a suitable free energy function.

1.1 The Ising Model

1.1.1 Definitions

Although some of the concepts presented in this chapter are fairly general, it is instructive to focus on a specific setting. We thus consider a model of N Ising spins $\{S_i\}_{i=1,\dots,N}$, with $S_i = \pm 1$. The energy of a spin configuration $\underline{S} = \{S_i\}$ is given by the Hamiltonian function

$$H[\underline{S}] = -\frac{1}{2}\sum_{ij} J_{ij} S_i S_j - \sum_i B_i S_i, \tag{1.1}$$

where $J_{ij} = J_{ji} \in \mathbb{R}$ denotes the (symmetric) exchange coupling between spins i and j (with $J_{ii} = 0, \forall i$), and $B_i \in \mathbb{R}$ denotes an external magnetic field acting on spin i. The factor $1/2$ in front of the exchange energy ensures that each pair ij is counted only once. The Hamiltonian in Eq. (1.1) summarises many cases of interest,

such as each spin interacting with all others (the 'fully connected' model) [69], or with a random subset of them (the 'Bethe lattice' or 'random graph' model) [251]. In general, the set of non-zero couplings defines the 'interaction graph'; its nodes are the spins and its edges are the pairs $\langle ij \rangle$, such that $J_{ij} \neq 0$. In the following, the neighbourhood of spin i, denoted by

$$\partial i = \{j : J_{ij} \neq 0\}, \tag{1.2}$$

is the set of all spins j that interact with spin i.

The ferromagnetic Ising model in d dimensions corresponds to a particular choice of interaction graph: a d-dimensional cubic lattice of unit spacing and linear size L, containing $N = L^d$ lattice points. Each spin i sits at a point $\mathbf{x}(i)$ of the lattice and interacts only with its nearest neighbours, located at unit distance away in each principal direction. Because there is a bijective correspondence between labels $i = 1, \dots, N$ and lattice points $\mathbf{x}$, both labels can be used equivalently. In the following, spins will thus be denoted either S_i or $S_{\mathbf{x}}$ depending on which labelling is more convenient. Each pair of nearest neighbour spins interact via a coupling $J_{ij} = J = \frac{1}{2d}$, and $J_{ij} = 0$ otherwise. In lattice notation, this corresponds to $J_{\mathbf{xy}} = \delta_{|\mathbf{x}-\mathbf{y}|,1}/(2d)$, where $|\mathbf{x}-\mathbf{y}|$ is the Euclidean distance between points $\mathbf{x}, \mathbf{y}$ and δ_{ab} is the Kronecker delta.

Because a cubic lattice has a boundary, one needs to specify boundary conditions. Three choices are commonly used [158].

- **Periodic boundary conditions** – Each face of the lattice is identified with its opposite face. In this case, the lattice is translationally invariant in all directions, and each spin has $2d$ nearest neighbours, corresponding to displacements in all possible directions on the lattice; hence, the size of ∂i is $2d$. In $d = 2$, the topology is that of a torus, generalised to a hypertorus in $d \geq 3$.
- **Open boundary conditions** – The lattice is considered isolated. In this case, the system is not translationally invariant. In particular, the spins on the boundary have fewer interactions than those in the bulk.
- **Frozen boundary conditions** – A layer of external spins is added to each face of the lattice. The external spins are frozen in prescribed positions: for example, they are all fixed to $+1$ or to -1. In this case, each spin has $2d$ nearest neighbours, but spins on the boundary interact with one frozen spin (or more, for those on cube edges), acting as an effective external magnetic field. Also in this case, the system is not translationally invariant.

Note that the overall magnitude of the couplings J only sets the energy and temperature scales and is therefore irrelevant for the properties of the model. The choice $J = \frac{1}{2d}$ guarantees that the exchange energy remains finite when $d \to \infty$, for any

value of N. For instance, with periodic boundary conditions, the fully magnetised spin configuration $\underline{1} = \{S_i = 1, \forall i\}$ has

$$H[\underline{1}] = -\sum_{\langle ij \rangle} J_{ij} - \sum_i B_i = -NJd - \sum_i B_i = -\frac{N}{2} - \sum_i B_i, \tag{1.3}$$

which explicitly shows that the exchange energy is finite for any N. Because the exchange energy remains of the same order as the entropy and the magnetic field energy, the model behaviour is interesting at finite temperature and magnetic field.[1]

1.1.2 Thermodynamic Free Energy and Observables

In equilibrium statistical mechanics, within the canonical ensemble, the probability of observing a spin configuration $\underline{S}$ is given by the Gibbs–Boltzmann distribution

$$P_{\rm GB}[\underline{S}] = \frac{e^{-\beta H[\underline{S}]}}{Z}, \tag{1.4}$$

where $Z = \sum_{\underline{S}} e^{-\beta H[\underline{S}]}$ is the partition function and $\beta = 1/T$ is the inverse temperature.[2] Solving the model amounts to computing the free energy of the system at temperature T:

$$F = -T \log Z = -T \log \sum_{\underline{S}} e^{\frac{\beta}{2} \sum_{ij} J_{ij} S_i S_j + \beta \sum_i B_i S_i}. \tag{1.5}$$

Note that F is extensive – i.e., proportional to N. In the following, capital letters are used for extensive thermodynamic quantities, while lowercase letters are used for intensive quantities that remain finite in the thermodynamic limit, $N \to \infty$ – e.g., $\mathrm{f} = F/N$. From the free energy, one can derive the statistical average of any observable, which we denote by brackets $\langle \bullet \rangle$. For example, the average energy is

$$U = \langle H[\underline{S}] \rangle = \sum_{\underline{S}} H[\underline{S}] \frac{e^{-\beta H[\underline{S}]}}{Z} = \frac{\partial(\beta F)}{\partial \beta}, \tag{1.6}$$

the entropy is

$$S = -\sum_{\underline{S}} P_{\rm GB}[\underline{S}] \log P_{\rm GB}[\underline{S}] = \beta(U - F), \tag{1.7}$$

[1] All the thermodynamic quantities are functions of T/J and B/J for dimensional reasons. Hence, different choices of J require rescaling both T and B with d to avoid a trivial behaviour of the model dominated by entropy or energy. Once T and B are properly rescaled, the result is invariant. For instance, the choice $J = 1$ leads to an exchange energy of order d and thus requires $T, B \propto d$.

[2] The Boltzmann constant k_B is fixed to unity throughout this book. In other words, T stands for $k_B T$ in such a way that temperatures are measured in the same units as energies. Similarly, entropy S stands for S/k_B and is thus adimensional.

the local magnetisation is

$$m_i = \langle S_i \rangle = \sum_{\underline{S}} S_i \frac{e^{-\beta H[\underline{S}]}}{Z} = -\frac{\partial F}{\partial B_i}, \tag{1.8}$$

and the local magnetic susceptibility is

$$\chi_{ij} = \frac{\partial m_i}{\partial B_j} = -\frac{\partial^2 F}{\partial B_i \partial B_j} = \beta(\langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle). \tag{1.9}$$

By construction, χ_{ij} is a positive matrix; i.e., all its eigenvalues are greater than or equal to zero.[3] The global magnetic susceptibility is the variation of the global magnetisation $m = N^{-1} \sum_i m_i$ with respect to a uniform magnetic field $B_i = B, \forall i$. Under a variation of the global field, $\mathrm{d}B$, all local fields change by $\mathrm{d}B_i = \mathrm{d}B$, hence $\mathrm{d}B_i/\mathrm{d}B = 1, \forall i$ and

$$\chi = \frac{\mathrm{d}m}{\mathrm{d}B} = \frac{1}{N} \sum_i \frac{\mathrm{d}m_i}{\mathrm{d}B} = \frac{1}{N} \sum_{ij} \frac{\partial m_i}{\partial B_j} \frac{\mathrm{d}B_j}{\mathrm{d}B} = \frac{1}{N} \sum_{ij} \chi_{ij}. \tag{1.10}$$

In general, any physical observable can be written as a linear combination of products of spin variables,[4]

$$\mathcal{O}[\underline{S}] = \sum_{n=0}^{N} \sum_{i_1 \cdots i_n} \mathcal{O}_{i_1 \cdots i_n} S_{i_1}, \ldots, S_{i_n}. \tag{1.11}$$

Therefore, in order to characterise all thermodynamic averages, it suffices to consider the average of spin products $\langle S_{i_1}, \ldots, S_{i_n} \rangle$, which can be computed as multiple derivatives of the free energy with respect to the relevant magnetic fields.

1.1.3 Free Energy as a Function of the Local Magnetisation

In order to investigate the high dimensional limit, it is convenient to introduce the free energy function $F[\underline{m}]$ which gives, roughly speaking, the free energy of a system constrained to have a prescribed set of local magnetisations $\underline{m} = \{m_i\}_{i=1,\ldots,N}$.

[3] Writing equivalently Eq. (1.9) as $\chi_{ij} = \beta \langle (S_i - m_i)(S_j - m_j) \rangle$, it follows, for any vector v, that

$$v^T \chi v = \sum_{ij} v_i \chi_{ij} v_j = \beta \sum_{ij} \langle v_i (S_i - m_i)(S_j - m_j) v_j \rangle = \beta \Big\langle \Big[\sum_i v_i (S_i - m_i) \Big]^2 \Big\rangle \geq 0.$$

This result holds in particular for the normalised eigenvectors of χ, for which $\chi v = \lambda v$, with λ the corresponding eigenvalue; hence, $v^T \chi v = \lambda v^T v = \lambda \geq 0$. This relationship proves that the eigenvalues are all positive.

[4] Equation (1.11) is easily justified by noting that $\mathcal{O}[\underline{S}]$ can take at most 2^N values, one for each spin configuration. The expression on the right-hand side of Eq. (1.11) contains $\binom{N}{n}$ coefficients for each n, which gives $\sum_{n=0}^{N} \binom{N}{n} = 2^N$ coefficients in total.

The local magnetisations are fixed by introducing a set of auxiliary local magnetic fields $\underline{b} = \{b_i\}_{i=1,\dots,N}$, and defining the corresponding auxiliary free energy:

$$\Omega[\underline{b}] = -T \log \sum_{\underline{S}} e^{-\beta H[\underline{S}] + \beta \sum_i b_i S_i}. \tag{1.12}$$

Note that the 'physical' magnetic fields B_i are included in $H[\underline{S}]$, while the auxiliary fields are denoted explicitly as we will use them to constrain the local magnetisations. Obviously, Eqs. (1.8) and (1.9) hold equivalently if one takes derivatives of $\Omega[\underline{b}]$ with respect to the auxiliary fields b_i. Because the matrix of its second derivatives is $-\chi_{ij}$, which is negative, $\Omega[\underline{b}]$ is necessarily a concave function. The Legendre transform of $\Omega[\underline{b}]$ defines $F[\underline{m}]$ as

$$\begin{aligned} -\beta F[\underline{m}] &= -\beta \max_{\underline{b}} \left[\Omega[\underline{b}] + \sum_i b_i m_i \right] \\ &= \min_{\underline{b}} \left[\log \sum_{\underline{S}} e^{-\beta H[\underline{S}] + \beta \sum_i b_i (S_i - m_i)} \right], \end{aligned} \tag{1.13}$$

which can be justified as follows. For any finite N, $\Omega[\underline{b}]$ is everywhere differentiable, and its concavity ensures that the maximum over $\underline{b}$ exists and is unique. In this case, the value $\underline{b} = \underline{b}[\underline{m}]$ that corresponds to the maximum is the unique solution of $\partial\Omega[\underline{b}]/\partial b_i = -m_i$; i.e., it is the set of local fields $b_i[\underline{m}]$ needed to enforce the magnetisations m_i. Once these values are computed, the value of $\Omega[\underline{b}[\underline{m}]]$ gives the free energy of the system with field $\underline{b}[\underline{m}]$. By subtracting the additional magnetic energy due to the external field, $-\sum_i b_i m_i$ in Eq. (1.13), one obtains the free energy $F[\underline{m}]$ of the system constrained to have local magnetisation $\underline{m}$. However, in the thermodynamic limit, $\Omega[\underline{b}]$ can develop singularities (in the vicinity of a phase transition) and become non-differentiable. This complicates the discussion of the Legendre transform, as will be detailed in Section 1.4.

Note that the derivative of $F[\underline{m}]$ (when it exists) is the auxiliary field $\underline{b}[\underline{m}]$ that corresponds to the maximum in Eq. (1.13):

$$b_i = \frac{\partial}{\partial m_i} F[\underline{m}], \qquad \frac{\partial b_i}{\partial m_j} = \frac{\partial^2 F[\underline{m}]}{\partial m_i \partial m_j} = (\chi^{-1})_{ij}. \tag{1.14}$$

The matrix χ^{-1} is positive, because χ is positive. $F[\underline{m}]$ is thus a convex function,[5] and the free energy $\Omega[\underline{b}]$ can be recovered as its inverse Legendre transform:

$$-\beta\Omega[\underline{b}] = -\beta \min_{\underline{m}} \left[F[\underline{m}] - \sum_i b_i m_i \right]. \tag{1.15}$$

[5] Note that the convexity of $F[\underline{m}]$ also directly follows from its definition, Eq. (1.13), because the maximum (over $\underline{b}$) of linear functions (of $\underline{m}$) is a convex function.

The stationarity condition implies that $\underline{m}[\underline{b}]$ is a solution of Eq. (1.14), and it must be a minimum because the second derivative of $F[\underline{m}]$ is positive. This result leads to an important observation: if there are no auxiliary fields, $b_i = 0$, then $\Omega[\underline{b} = \underline{0}] = F$ is equal to the thermodynamic free energy, as defined in Eq. (1.5). We obtain from Eq. (1.15) that

$$F = \min_{\underline{m}} F[\underline{m}]. \tag{1.16}$$

The thermodynamic free energy thus corresponds to the minimum of $F[\underline{m}]$ over all possible sets of local magnetisations $\underline{m}$, and the set of local magnetisations that achieves the minimum in Eq. (1.16) corresponds to the equilibrium magnetisations in absence of any auxiliary field.

1.2 Large Dimension Expansion for the Ising Model

The function $F[\underline{m}]$ contains a lot of information but is unfortunately impossible to compute explicitly for the Ising model when d is finite. One can nonetheless try to obtain information by constructing a perturbative expansion. The simplest perturbative expansion of $F[\underline{m}]$ is the high-temperature expansion [296], which here also coincides with the large d expansion [163], as shown in this section.

1.2.1 Infinite Temperature

At infinite temperature ($\beta = 0$), the Gibbs–Boltzmann probability distribution $e^{-\beta H[S]}/Z$ is uniform over all spin configurations. In order to fix the magnetisations, we need strong magnetic fields; we thus rescale the fields b_i introducing $\lambda_i = \beta b_i$. Because the entropy S at infinite temperature is finite, while the energy U vanishes, the free energy $F = U - TS$ diverges proportionally to $T = 1/\beta$ when $T \to \infty$. It is thus better to consider $A = -\beta F = S - \beta U$ which remains finite even at $\beta = 0$. With these rescalings, Eq. (1.12) becomes

$$A_0[\underline{\lambda}] = -\beta \Omega_0[\underline{b}] = \log \sum_{\underline{S}} e^{\sum_i \lambda_i S_i} = \sum_i \log[2\cosh(\lambda_i)], \tag{1.17}$$

where the suffix 0 highlights that this is the zeroth order ($\beta = 0$) expression. The free energy is an analytic function of the auxiliary fields, as obviously there is no phase transition at infinite temperature. The Legendre transform can thus be computed by differentiation. The condition that determines λ_i is

$$m_i = -\frac{\partial \Omega_0[\underline{b}]}{\partial b_i} = \frac{\partial A_0[\underline{\lambda}]}{\partial \lambda_i} = \tanh(\lambda_i), \tag{1.18}$$

and, according to Eq. (1.13),

$$\begin{aligned} -\beta F_0[\underline{m}] &= A_0[\underline{\lambda}[\underline{m}]] - \sum_i \lambda_i[\underline{m}]m_i \\ &= \sum_i \{\log[2\cosh(\lambda_i[\underline{m}])] - \lambda_i[\underline{m}]m_i\} = \sum_i s_0(m_i), \\ s_0(m_i) &= -\left(\frac{1+m_i}{2}\log\frac{1+m_i}{2} + \frac{1-m_i}{2}\log\frac{1-m_i}{2}\right). \end{aligned} \tag{1.19}$$

The function $s_0(m)$ is the entropy of a single spin constrained to have magnetisation[6] m; at infinite temperature, spins are independent, and the total entropy – which coincides with $-\beta F_0[\underline{m}]$ – is the sum of the single-spin entropies.

The first small β correction can be computed easily. We only sketch here the derivation. First, one expands $A[\underline{\lambda}]$ in powers of β as follows:

$$\begin{aligned} A[\underline{\lambda}] &= \log \sum_{\underline{S}} e^{\sum_i \lambda_i S_i} e^{-\beta H[\underline{S}]} = \log \sum_{\underline{S}} e^{\sum_i \lambda_i S_i}(1 - \beta H[\underline{S}]) + O(\beta^2) \\ &= A_0[\lambda] - \beta \frac{\sum_{\underline{S}} e^{\sum_i \lambda_i S_i} H[\underline{S}]}{\sum_{\underline{S}} e^{\sum_i \lambda_i S_i}} + O(\beta^2). \end{aligned} \tag{1.20}$$

The first correction is then given by the average of $H[\underline{S}]$ over independent spins subjected to magnetic fields λ_i. The average of S_i is $\tanh(\lambda_i)$, and, therefore,[7] the average of $H[\underline{S}]$ is $H[\tanh(\underline{\lambda})]$, where $\tanh(\underline{\lambda}) = \{\tanh(\lambda_i)\}_{i=1,\dots,N}$. We thus have $A[\underline{\lambda}] = A_0[\underline{\lambda}] + \beta A_1[\underline{\lambda}]$, with $A_1[\underline{\lambda}] = -H[\tanh(\underline{\lambda})]$. The equation for $\underline{\lambda}$ then becomes $m_i = \tanh(\lambda_i) + \beta\partial A_1[\underline{\lambda}]/\partial\lambda_i$, and $\lambda_i = \lambda_i^0 + \beta\lambda_i^1$ with $\lambda_i^0 = \operatorname{atanh}(m_i)$. The Legendre transform is obtained by writing

$$\begin{aligned} -\beta F[\underline{m}] &= A_0[\underline{\lambda}^0 + \beta\underline{\lambda}^1] + \beta A_1[\underline{\lambda}^0 + \beta\underline{\lambda}^1] - \sum_i (\lambda_i^0 + \beta\lambda_i^1)m_i \\ &= A_0[\underline{\lambda}^0] + \beta A_1[\underline{\lambda}^0] - \sum_i \lambda_i^0 m_i = \sum_i s_0(m_i) - \beta H[\underline{m}]. \end{aligned} \tag{1.21}$$

[6] Because an Ising spin has only two states, the probability distribution $p(S)$ of a single spin is expressed by two real numbers $p(1)$ and $p(-1)$ satisfying $p(1) + p(-1) = 1$. It is thus specified by one real number, which can be conveniently chosen to be the average magnetisation of the spin, $m = \sum_S Sp(S)$; hence, $p(S) = (1 + mS)/2$. With this choice, the single spin entropy

$$s_0(m) = -p(1)\log p(1) - p(-1)\log p(-1)$$

coincides with Eq. (1.19).

[7] Note that this is true only if $H[\underline{S}]$ is explicitly written as a linear combination of products of spin variables, which is true for our reference expression Eq. (1.1). Any general spin Hamiltonian can also be written in this form, using Eq. (1.11).

Note that the contributions of $\underline{\lambda}^1$ have disappeared from Eq. (1.21) for the following reasons. In A_1, $\underline{\lambda}^1$ can be eliminated because it gives contributions of order β^2. In the remaining terms, $A_0[\underline{\lambda}] - \sum_i \lambda_i m_i$, the derivative with respect to $\underline{\lambda}$ vanishes identically due to the Legendre transform condition, and, therefore, at first order, the correction to $\underline{\lambda}$ disappears. Finally, the terms $A_0[\underline{\lambda}^0] - \sum_i \lambda_i^0 m_i$ give the infinite-temperature result, $\sum_i s_0(m_i)$, while the correction is $\beta A_1[\underline{\lambda}^0]$ $= -\beta H[\tanh(\underline{\lambda}^0)] = -\beta H[\underline{m}]$.

1.2.2 High-Temperature Expansion

At first order in β, the free energy function $F[\underline{m}] = H[\underline{m}] - T \sum_i s_0(m_i)$ is thus given by the free energy of independent spins with magnetisations $\underline{m}$, as shown by Eq. (1.21). The computation of higher-order corrections in β goes along the same lines as the first order correction, as described in [163]. At third order, for instance, the result is

$$\begin{aligned} -\beta F[\underline{m}] = &- \sum_i \left(\frac{1+m_i}{2} \log \frac{1+m_i}{2} + \frac{1-m_i}{2} \log \frac{1-m_i}{2} \right) \\ &+ \beta \frac{1}{2} \sum_{ij} J_{ij} m_i m_j + \beta \sum_i B_i m_i \\ &+ \frac{\beta^2}{4} \sum_{ij} J_{ij}^2 (1-m_i^2)(1-m_j^2) \\ &+ \frac{\beta^3}{6} \left[2 \sum_{ij} J_{ij}^3 m_i (1-m_i^2) m_j (1-m_j^2) \right. \\ &\left. + \sum_{ijk} J_{ij} J_{ik} J_{jk} (1-m_i^2)(1-m_j^2)(1-m_k^2) \right] + O(\beta^4), \end{aligned} \tag{1.22}$$

and

$$\begin{aligned} \beta b_i[\underline{m}] = \frac{\partial \beta F[\underline{m}]}{\partial m_i} &= \operatorname{atanh}(m_i) - \beta \left[\sum_j J_{ij} m_j + B_i \right] \\ &+ \beta^2 m_i \sum_j J_{ij}^2 (1-m_j^2) + O(\beta^3). \end{aligned} \tag{1.23}$$

Imposing the absence of auxiliary fields, $b_i = 0$, is equivalent to minimising $F[\underline{m}]$. In this case, one obtains the so-called Thouless–Anderson–Palmer (TAP) equations [342],

$$m_i = \tanh \beta h_i,$$
$$h_i = B_i + \sum_j J_{ij} m_j - \beta m_i \sum_j J_{ij}^2 (1 - m_j^2) + O(\beta^2), \tag{1.24}$$

where h_i is the effective magnetic field provided by the spins neighbouring site i. Note that the TAP Eq. (1.24) are examples of a general class of mean field equations for disordered systems. These equations can be derived through high-temperature expansions, as discussed here, or alternatively via probabilistic methods called the 'cavity method' or 'belief propagation' [251, 254, 372].

1.2.3 Large Dimension Expansion

The high-temperature expansion in Eq. (1.22) can be specialised to the ferromagnetic d-dimensional Ising model, with periodic boundary conditions and with a uniform external magnetic field $B_i = B$. In this case, the system is translationally invariant. It is, therefore, reasonable to expect that the free energy minimum is realised by a uniform magnetisation $m_i = m$. In the following, the free energy per spin corresponding to a uniform magnetisation, also called 'potential', is denoted[8] by $\mathrm{v}(m)$. Equation (1.22) gives

$$\mathrm{v}(m) = \frac{F[\{m_i = m, \forall i\}]}{N} = -T s_0(m) - \frac{1}{2} m^2 - Bm - \frac{\beta}{8d}(1 - m^2)^2 - \frac{\beta^2}{12 d^2} m^2 (1 - m^2)^2 + O(\beta^3). \tag{1.25}$$

With the choice of coupling scale $J = 1/(2d)$, the first three terms, which represent entropy, exchange energy and magnetic field energy, respectively, remain finite for $d \to \infty$, as discussed in Section 1.1.1. The correction terms form a series in powers of $1/d$: the high-temperature (small β) expansion can be used to construct the

[8] To clarify the rationale behind this notation, it is convenient to anticipate here briefly a few notions that will be clarified later in the chapter. In a coarse grained representation the spatial dependent magnetisation profile is $m(\mathbf{x})$. The thermodynamic free energy $F(m)$ corresponding to a global magnetisation m is the minimum of $F[\underline{m}] \sim F[m(\mathbf{x})]$ over all configurations $m(\mathbf{x})$ such that $m = N^{-1} \sum_{i=1}^N m_i \sim V^{-1} \int \mathrm{d}\mathbf{x} m(\mathbf{x})$. Although the minimum is often realised by uniform configurations, this is not always the case. Examples are antiferromagnets or systems in the phase coexistence region. In a coarse grained representation, corresponding to Landau theory, one can approximate

$$F[m(\mathbf{x})] \sim \int \mathrm{d}\mathbf{x} \{c[\nabla m(\mathbf{x})]^2 + \mathrm{v}[m(\mathbf{x})]\},$$

where $\mathrm{v}(m)$ is the free energy of a uniform configuration, which then provides a 'potential' term in the total free energy. If $\mathrm{v}(m)$ has a unique minimum, then the profile is uniform; the gradient term disappears and $F(m) = N\mathrm{v}(m)$. Otherwise, phase coexistence can lead to $F(m) < N\mathrm{v}(m)$ for some values of m. See Section 1.4 for a more detailed discussion.

large d (small $1/d$) expansion.[9] The same result holds also in presence of disorder, although the proper scaling of the couplings might then differ.[10]

It is important to stress that while the true function $F[\underline{m}]$ is guaranteed to be convex, any truncation at a finite order of its high-temperature expansion in Eq. (1.22) does not necessarily share this property. For example, it is very easy to check that Eq. (1.25), truncated at any order in β, is not a convex function of m if the temperature is low enough. This point will be further discussed in Section 1.4.

1.3 Second-Order Phase Transition of the Ising Ferromagnet

The large d expansion in Eq. (1.25) can be used to investigate the ferromagnetic phase transition, a second-order phase transition that is observed in many magnetic materials and is well described by the Ising model [11, 69, 282]. In this section, we discuss the nature of this transition in the limit $d \to \infty$, the corrections in $1/d$ and the nature of spin–spin correlations around the transition.

1.3.1 Mean Field: The Curie–Weiss Model

For a uniform magnetisation profile $m_i = m$, keeping only the terms that remain finite for $d \to \infty$, the potential is

$$\mathrm{v}(m) = -T s_0(m) - \frac{1}{2}m^2 - Bm, \tag{1.26}$$

and the TAP equations that determine m simplify to reproduce the mean field result,

$$m = \tanh[\beta(B + m)]. \tag{1.27}$$

The global magnetic susceptibility $\chi = \mathrm{d}m/\mathrm{d}B$ is then obtained by differentiating Eq. (1.27) with respect to B:

$$\frac{\mathrm{d}m}{\mathrm{d}B} = \{1 + \tanh[\beta(B + m)]^2\}\beta\left(1 + \frac{\mathrm{d}m}{\mathrm{d}B}\right) \;\Rightarrow\; \chi = \frac{\beta(1 + m^2)}{1 - \beta(1 + m^2)}. \tag{1.28}$$

Similar expressions have been derived by Curie and Weiss under a mean field approximation, and for this reason, the infinite-dimensional Ising model is also known as the Curie–Weiss model [69]. In fact, Eq. (1.27) can be equivalently obtained by assuming that the neighbours of a given spin i (whose set we denote ∂i, see Eq. (1.2)) are uncorrelated and replacing the spins by their average. The local field, $h_i = B_i + \frac{1}{2d}\sum_{j\in\partial i} S_j$, is then on average equal to $h_i = B + m$. The magnetisation of an isolated spin in a field h_i is $m = \tanh(\beta h_i)$, from which one

[9] See [163, figure 3] for additional correction terms.
[10] For instance, if J_{ij} are independent Gaussian variables with zero mean, the natural scaling is $J_{ij} \propto 1/\sqrt{d}$.