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EISENSTEIN SERIES AND AUTOMORPHIC REPRESENTATIONS

This introduction to automorphic forms on adelic groups $G(\mathbb{A})$ emphasises the role of representation theory. The exposition is driven by examples and collects and extends many results scattered throughout the literature, in particular the Langlands constant term formula for Eisenstein series on $G(\mathbb{A})$ as well as the Casselman–Shalika formula for the p-adic spherical Whittaker function. This book also covers more advanced topics such as spherical Hecke algebras and automorphic L-functions.

Many of these mathematical results have natural interpretations in string theory, and so some basic concepts of string theory are introduced with an emphasis on connections with automorphic forms. Throughout the book special attention is paid to small automorphic representations which are of particular importance in string theory, but are also of independent mathematical interest. Numerous open questions and conjectures, partially motivated by physics, are included to prompt the reader's own research.

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Eisenstein Series and Automorphic Representations

with Applications in String Theory

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Meinen ersten Schritten in der Wissenschaft
–P.F.

To my beloved family
–H.P.A.G.

Für meine liebe Mutter
–A.K.

Till min underbara familj
–D.P.

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Preface

This book is the result of our endeavour to give a comprehensive presentation of the theory of automorphic representations and the structure of Fourier expansions of automorphic forms with a particular emphasis on adelic methods and Eisenstein series. Our intention is also to open up a channel of communication between mathematicians and physicists, in particular string theorists, interested in these topics. Many of the results presented herein exist already in the literature and we benefited greatly from [447, 116, 291, 292, 569, 164, 290, 598, 599]; our exposition differs, however, from these references in the selection of material and also in highlighting the connection to theoretical physics. Several new results and examples are included as well; in particular, we provide many techniques for working out aspects of the Fourier expansion of Eisenstein series and studying automorphic representations of small functional dimension.

This book is divided into three parts:

PART ONE Automorphic Representations
contains the mathematical backbone of the book, including adeles, Lie theory, Eisenstein series, automorphic representations, Fourier expansions and Hecke theory

PART TWO Applications in String Theory
contains the salient features of string theory to understand the way automorphic forms and representations arise; including scattering amplitudes, perturbative and non-perturbative aspects and supersymmetry

PART THREE Advanced Topics
contains a mix of topics from mathematics and physics, often situated at the interface and highlighting ideas and open problems that can or should be transferred from one field to the other

Many chapters in PART ONE and PART TWO contain sections that are marked with an asterisk and these are more advanced and not strictly necessary for the development of the theory. PART THREE as a whole is advanced by definition and includes many appetisers for current research.

Section 1.4 contains a detailed reader's guide to this book and we refrain from duplicating this summary here. The labelling of theorems, definitions, propositions, remarks and examples is done within each chapter without reference to the (sub)section they are in. Two separate lists, one for examples and one for everything else, are provided at the beginning of the book.

Our understanding of the material presented here was greatly facilitated by numerous discussions with colleagues both from the mathematics community and from the string theory community. We are especially indebted to Guillaume Bossard, David Ginzburg, Joseph Hundley, Stephen D. Miller, Hermann Nicolai, Bengt E.W. Nilsson and Boris Pioline for many clarifying and stimulating discussions and collaborations over many years.

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We would be very grateful to learn of any omissions and mistakes that we have made unintentionally. A list of errata will be maintained on the website

`www.aei.mpg.de/~axkl/AutomorphicBook`

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Daniel Persson