ULTRAMETRIC PSEUDODIFFERENTIAL EQUATIONS AND APPLICATIONS

Starting from physical motivations and leading to practical applications, this book provides an interdisciplinary perspective on the cutting edge of ultrametric pseudodifferential equations. It shows the ways in which these equations link different fields, including mathematics, engineering, and geophysics. In particular, the authors provide a detailed explanation of the geophysical applications of p-adic diffusion equations useful when modeling the flows of liquids through porous rock. p-Adic wavelets theory and p-adic pseudodifferential equations are also presented, along with their connections to mathematical physics, representation theory, the physics of disordered systems, probability, number theory, and p-adic dynamical systems.

Material that was previously spread across many articles in journals of many different fields is brought together here, including recent work on the van der Put series technique. This book provides an excellent snapshot of the fascinating field of ultrametric pseudodifferential equations, including their emerging applications and currently unsolved problems.

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Encyclopedia of Mathematics and its Applications

Ultrametric Pseudodifferential Equations and Applications

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Andrei Yu. Khrennikov , Sergei V. Kozyrev , W. A. Zúñiga-Galindo
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Preface

The present book aims to provide an interdisciplinary perspective of the state of the art of the theory of ultrametric equations and its applications, starting from physical motivations and applications of the ultrametric geometry, and covering connections with probability, functional analysis, number theory, etc. in a novel form. In recent years the connections between non-Archimedean mathematics (mainly analysis) and mathematical physics have received a lot of attention, see e.g. [53]–[60], [63], [90], [90], [132]–[137], [164]–[166], [168], [190], [191], [220]–[228], [322]–[328], [336], [346]–[350], [366], [373], [413]–[411], [423]–[435] and the references therein. All these developments have been motivated by two physical ideas. The first is the conjecture (due to Igor Volovich) in particle physics that at Planck distances space-time has a non-Archimedean structure, see e.g. [438]–[435], [413], [412]. The second idea comes from statistical physics, more precisely, in connection with models describing relaxation in glasses, macromolecules, and proteins. It has been proposed that the non-exponential nature of those relaxations is a consequence of a hierarchical structure of the state space which can in turn be related to p-adic structures. Giorgio Parisi introduced the idea of hierarchy for spin glasses (disordered magnetics) in a more precise form in 1979, then the idea was extended to other physical problems and combinatorial optimization problems, see [336]. Then in the 1980s effects of slow non-exponential relaxation and aging were observed in deeply frozen proteins, implying the occurrence of a glass transition similar to that in spin glasses. Thus in the middle of the 1980s the idea of using ultrametric spaces to describe the states of complex biological systems, which naturally possess a hierarchical structure, emerged in the works of Frauenfelder, Parisi, Stain, and among others see e.g. [164]. In protein physics, it is regarded as one of the most profound ideas put forward to explain the nature of distinctive attributes of life.

For replica symmetry breaking in spin glasses the p-adic models were proposed independently by Avetisov *et al.* [53] and Parisi and Sourlas [373]. The idea of using p-adic diffusion equation to describe protein relaxation was proposed in [53].

From a mathematical point of view, in these models the time-evolution of a complex system is described by a p-adic master equation (a parabolic-type pseudodifferential equation) which controls the time-evolution of a transition function of a Markov process on an ultrametric space, and this stochastic process is used to describe the dynamics of the system in the space of configurational states which is approximated by an ultrametric space ($\mathbb{Q}_p$). This is a central motivation for developing a theory of ultrametric reaction–diffusion equations or, more generally, a theory of pseudodifferential equations on ultrametric spaces.

The simplest ultrametric diffusion equation is the one-dimensional p-adic heat equation. This equation was introduced in the book of Vladimirov, Volovich, and Zelenov [434, Section XVI]. Kochubei [275, Chapters 4 and 5] presented a general theory for one-dimensional parabolic-type pseudodifferential equations with variable coefficients, whose fundamental solutions are transition density functions for Markov processes in the p-adic line, see also [11], [12], [104], [101], [386], [464], [411]. A p-adic diffusion equation was also considered by Albeverio and Karwowski [10]–[11]. Zúñiga-Galindo and his collaborators have developed a very general theory of linear pseudodifferential equations, based on the work of Kochubei, over p-adics and adeles, see [470]. At this point it is important to mention the differences between [470] and this book. The book [470] was written from the perspective of "pure mathematics," while this book has been written from an interdisciplinary perspective. There is a small intersection, namely the material presented in Sections 8.1–8.3, which corresponds to some basic results on p-adic parabolic-type equations and the associated Markov processes; this material is summarized here without proofs.

The tree-like structure of configuration spaces was widely used in applications to cognitive science and psychology, see, e.g., the pioneering works of Khrennikov [222], [223]; see also [141], [14]. Recently Khrennikov and Oleschko proposed using this class of configuration spaces in geology [253], [252]. This is a new area of research and very important for applications, especially because of the possibility of being able to couple the output of theoretical modeling with applied petroleum research (performed by the research team of Oleschko working on the Mexican oil fields). For the moment, only the first steps in this direction have been taken.

This book does not enlighten the reader concerning advanced research devoted to the models of mathematical physics with p-adic-valued wave functions (in particular, p-adic-valued probabilities), see [222] for details. We present only some results about the theory of p-adic dynamical systems, concerning iterations of maps in the fields of p-adic numbers. The development of this theory was partially motivated by mathematical physics, but later this theory was mainly explored in applications to modeling of cognition, see e.g. [222], [223], [141], [14], [20] and in cryptography, see e.g. [35], [39].

The book is organized as follows. In Chapter 1, we review, without proofs, the basic definitions and results on p-adic functional analysis of complex-valued functions of p-adic arguments, for an in-depth discussion of these results, the reader may consult [18], [402], [434]. Chapter 2 aims to present the essential ideas of ultrametrics in connection with clustering and trees. The material presented includes affine Bruhat–Tits buildings and multiclustering, groups acting on trees, and the Vladimirov operator. Chapter 3 is dedicated to the theory of p-adic wavelets and its applications. This chapter presents an in-depth discussion of p-adic multiresolution analysis and wavelet techniques for solving several types of general ultrametric equations. In addition, connections with mathematical physics and representation theory (the theory of coherent states) are also discussed. This material is not covered in references such as [18]. Chapter 4 aims to give a short review of some applications of p-adic and more general ultrametric methods in the statistical physics of disordered systems, dynamics of macromolecules, and genetics. A well-known and accepted scientific paradigm in the physics of complex systems (such as glasses and proteins) asserts that the dynamics of a large class of complex systems is described as a random walk on a complex energy landscape, see e.g. [164]–[166], [440], and [294, and references therein]. A landscape is a continuous real-valued function that represents the energy of a system. The term complex landscape means that the energy function has many local minima. In the case of complex landscapes, in which there are many local minima, a "simplification method" called interbasin kinetics is applied. The idea is to study the kinetics generated by transitions between groups of states (basins). A key idea is that the dynamics on a complex energy landscape is approximated by a family of Arrhenius transitions between local energy minima. Moreover, the set of local minima and transition states between the minima is given by a "disconnectivity graph" of basins (a tree) and by functions on this graph that describe the distributions of energies of the minima and activation energies of the transition states. The p-adic models introduced by Avetisov, Kozyrev *et al.* have master equations of the following form:

$$\frac{\partial f(x,t)}{\partial t} = \int_{\mathbb{Q}_p} [w(x|y)f(y,t) - w(y|x.)f(x,t)]dy, \tag{1}$$

where $x \in \mathbb{Q}_p, t \geq 0$. The function $f(x,t)\colon \mathbb{Q}_p \times \mathbb{R}_+ \to \mathbb{R}_+$ is a probability density distribution, so $\int_B f(x,t)dx$ is the probability of finding the system in a domain $B \subset \mathbb{Q}_p$ at the instant t. The function $w(x|y)\colon \mathbb{Q}_p \times \mathbb{Q}_p \to \mathbb{R}_+$ is the probability rate of the transition from state y to state x per unit of time.

In Chapter 5, we give applications of wavelet techniques for solving certain integral equations, and also applications to the construction of the p-adic one-dimensional version of Brownian motion.

In Chapter 6, we present a new conceptual approach for modeling of fluid flows in random porous media based on explicit exploration of the tree-like geometry of complex capillary networks. Such patterns can be represented mathematically as ultrametric spaces and the dynamics of fluids by ultrametric diffusion. In this model the porous background is treated as the environment contributing to the coefficients of evolutionary equations. For the simplest trees, these equations are significantly less complicated than those with fractional differential operators which are commonly applied in geological studies looking for some fractional analogs to conventional Euclidean space but with anomalous scaling and diffusion properties. The systems of ultrametric reaction–diffusion equations can be used to model the process of extraction of oil from an extended network of capillaries. This process is especially important for the design of oil recovery programs and especially for the selection of *enhanced oil recovery* (EOR) methods, where the fluid flow from the solid matrix is stimulated. In Chapter 6 a new non-linear p-adic pseudodifferential equation, which is the non-Archimedean counterpart of the *porous medium equation*, is introduced.

Chapter 7 describes recent developments in p-adic dynamical systems (see the monographs [222], [35]) and their connections with cryptography. Discrete dynamical systems based on iterations of functions belonging to the special functional class, namely, *1-Lipschitz functions*, are considered. The importance of this class for the theory of p-adic dynamical systems was emphasized in a series of pioneering works by V. Anashin [31], [32], [33]. Then some interesting results about such discrete dynamics were obtained in joint works by V. Anashin, A. Khrennikov, and E. Yurova, see, e.g., [34], [35], [452].

Chapter 8 has two goals. The first is to present general results for a large class of pseudodifferential equations, which contains equations of type (1). These equations are related to models of complex systems. In the second part, we introduce a new class of non-linear p-adic pseudodifferential equations. Chapter 9 is dedicated to the study of general p-adic diffusion equations driven by Gaussian noise.

Chapter 10 aims to present the basic results about the Sobolev-type spaces over $\mathbb{Q}_p^N$ and to show the existence of fundamental solutions for pseudodifferential equations over these spaces. We consider two types of spaces, denoted $\mathcal{H}_\infty$ and $\mathcal{W}_\infty$. Both spaces are countably Hilbert nuclear spaces, with $\mathcal{W}_\infty$ continuously embedded in $\mathcal{W}_\infty$. These spaces are invariant under the action of a large class of pseudodifferential operators. The spaces $\mathcal{H}_\infty$ were introduced by Zúñiga-Galindo in [472]. In the spaces $\mathcal{W}_\infty$ we show the existence of fundamental solutions for pseudodifferential operators whose symbols involve general polynomials. This result is the non-Archimedean counterpart of Hörmander's solution of the problem of the division of a distribution by a polynomial, see [202], [316]. We also summarize the results of [471], without proofs. In this work the existence of fundamental solutions for pseudodifferential equations using local zeta functions is established in the spaces $\mathcal{H}_\infty$.

In Chapter 11 we present a new class of non-Archimedean Euclidean quantum fields, in arbitrary dimension, which are constructed as solutions of certain covariant p-adic stochastic pseudodifferential equations (SPDEs), by using techniques of white-noise calculus. The connection between quantum fields and SPDEs has been studied intensively in the Archimedean setting, see e.g. [9]–[30] and the references therein. A massive non-Archimedean field $\boldsymbol{\Phi}$ is a random field parametrized by $\mathcal{H}_\infty\left(\mathbb{Q}_p^N;\mathbb{R}\right)$, the nuclear countably Hilbert spaces introduced in Chapter 10. Heuristically, $\boldsymbol{\Phi}$ is the solution of $\left(\boldsymbol{L}_\alpha + m^2\right)\boldsymbol{\Phi} = \digamma$, where $\boldsymbol{L}_\alpha$ is a pseudodifferential operator, $m > 0$, and $\digamma$ is a generalized Lévy noise. This type of noise is introduced in this chapter. Finally, as an application,we give a general construction of a p-adic Brownian sheet on $\mathbb{Q}_p^N$.

In Chapter 12, we commence the study of p-adic spectral zeta functions. In the real setting, the spectral zeta function attached to the Laplacian (under a suitable hypothesis) is the Riemann zeta function. This spectral zeta function is studied by using the techniques of heat equations. There are many types of p-adic heat equation, and thus many types of p-adic Laplacian. It is natural to study the spectral zeta functions of these p-adic Laplacians. Of course there are very serious arithmetical motivations for this study. In Chapter 12, we study heat traces and spectral zeta functions attached to certain p-adic Laplacians, like the ones introduced in Chapter 8, which are denoted $\boldsymbol{A}_\beta$. By using an approach inspired by the work of Minakshisundaram and Pleijel, see [340]–[342], we find a formula for the trace of the semigroup $e^{-t\boldsymbol{A}_\beta}$ acting on the space of square integrable functions supported on the unit ball with average zero. The trace of $e^{-t\boldsymbol{A}_\beta}$ is a p-adic oscillatory integral of Laplace–type. We do not know the exact asymptotics of this integral as t tends to infinity, however, we obtain a good estimation for its behavior at infinity.

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