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$p$-Adic Analysis: Essential Ideas and Results

In this chapter, we present, without proofs, the essential aspects of, and basic results on, $p$-adic functional analysis needed in the book. For a detailed exposition on $p$-adic analysis the reader may consult [18], [402], [434].

1.1 The Field of $p$-Adic Numbers

Throughout this book $p$ will denote a prime number. The field of $p$-adic numbers $\mathbb{Q}_p$ is defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the $p$-adic norm $|\cdot|_p$, which is defined as

$$|x|_p = \begin{cases} 0 & \text{if } x = 0 \\ p^{-\gamma} & \text{if } x = p^\gamma \frac{a}{b}, \end{cases}$$

where $a$ and $b$ are integers coprime with $p$. The integer $\gamma := \text{ord}(x)$, with $\text{ord}(0) := +\infty$, is called the $p$-adic order of $x$. We extend the $p$-adic norm to $\mathbb{Q}_p^N$ by taking

$$||x||_p := \max_{1 \leq i \leq N} |x_i|_p,$$

for $x = (x_1, \ldots, x_N) \in \mathbb{Q}_p^N$.

We define $\text{ord}(x) = \min_{1 \leq i \leq N} \{\text{ord}(x_i)\}$, then $||x||_p = p^{-\text{ord}(x)}$. The norm $|| \cdot ||_p$ satisfies

$$\|x + y\|_p \leq \max(\|x\|_p, \|y\|_p),$$

the strong triangle inequality. The metric space $(\mathbb{Q}_p^N, || \cdot ||_p)$ is a complete ultrametric space. As a topological space $\mathbb{Q}_p$ is homeomorphic to a Cantor-like subset of the real line, see e.g. [18], [434]. In Chapter 2 only, we will work with more general norms, namely, norms of type

$$\mathcal{N}_{q_1,\ldots,q_N}(x) = \max_{1 \leq i \leq N} q_i |x_i|_p,$$

where the $q_1, \ldots, q_N$ are fixed positive numbers.
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\textit{p-Adic Analysis: Essential Ideas and Results}

Any \( p \)-adic number \( x \neq 0 \) has a unique expansion of the form

\[
x = p^\text{ord}(x) \sum_{j=0}^{\infty} x_j p^j,
\]

where \( x_j \in \{0,1,2,\ldots,p-1\} \) and \( x_0 \neq 0 \). By using this expansion, we define the \textit{fractional part} of \( x \in \mathbb{Q}_p \), denoted \( \{x\}_p \), as the rational number

\[
\{x\}_p = \begin{cases} 0 & \text{if } x = 0 \text{ or ord}(x) \geq 0 \\ p^{\text{ord}(x)} \sum_{j=0}^{-\text{ord}(x)-1} x_j p^j & \text{if ord}(x) < 0. 
\end{cases}
\]

In addition, any \( p \)-adic number \( x \neq 0 \) can be represented uniquely as \( x = p^{\text{ord}(x)} \chi(x) \) with \( |\chi(x)|_p = 1 \); \( \chi(x) \) is called the \textit{angular component} of \( x \).

### 1.1.1 Additive Characters

Set \( \chi_p(y) := \exp(2\pi i(y)_p) \) for \( y \in \mathbb{Q}_p \). The map \( \chi_p(\cdot) \) is an additive character on \( \mathbb{Q}_p \), i.e. a continuous map from \( (\mathbb{Q}_p,+) \) into \( S \) (the unit circle considered as a multiplicative group) satisfying \( \chi_p(x_0 + x_1) = \chi_p(x_0)\chi_p(x_1) \), \( x_0, x_1 \in \mathbb{Q}_p \). We notice that \( \chi_p \) satisfies the following relations:

\[
\begin{align*}
\chi_p(0) &= 1, \\
\chi_p(-x) &= \overline{\chi_p(x)} = \chi_p^{-1}(x), \\
\chi_p(mx) &= \chi_p^m(x), \quad m \in \mathbb{Z}.
\end{align*}
\]

The additive characters of \( \mathbb{Q}_p \) form an Abelian group which is isomorphic to \( (\mathbb{Q}_p,+) \); the isomorphism is given by \( \xi \rightarrow \chi_p(\xi x) \), see e.g. [18, Section 2.3]. We will call \( \chi_p(\cdot) \) the \textit{standard additive character} of \( \mathbb{Q}_p \).

### 1.2 Topology of \( \mathbb{Q}_p^N \)

For \( r \in \mathbb{Z} \), denote by \( B_r^N(a) = \{x \in \mathbb{Q}_p^N; |x - a|_p \leq p^r\} \) the \textit{ball of radius } \( p^r \) \textit{with center at} \( a = (a_1, \ldots, a_N) \in \mathbb{Q}_p^N \), and take \( B_0^N(0) := B_0^N \). Note that \( B_r^N(a) = B_r(a_1) \times \cdots \times B_r(a_N) \), where \( B_r(a_i) := \{x \in \mathbb{Q}_p; |x_i - a_i|_p \leq p^r\} \) is the one-dimensional ball of radius \( p^r \) with center at \( a_i \in \mathbb{Q}_p \). The ball \( B_0^N \) equals the product of \( N \) copies of \( \mathbb{Z}_p \), the \textit{ring of } \( p \)-adic integers. We also denote by \( S_r^N(a) = \{x \in \mathbb{Q}_p^N; |x - a|_p = p^r\} \) the \textit{sphere of radius } \( p^r \) \textit{with center at} \( a = (a_1, \ldots, a_N) \in \mathbb{Q}_p^N \), and take \( S_0^N(0) := S_0^N \). We notice that \( S_0^N = \mathbb{Z}_p^N \) (the group of units of \( \mathbb{Z}_p \)), but \( (\mathbb{Z}_p^N)^* \subseteq S_0^N \). The balls and spheres are both open and closed subsets in \( \mathbb{Q}_p^N \). In addition, two balls in \( \mathbb{Q}_p^N \) are either disjoint or one is contained in the other.

As a topological space \( (\mathbb{Q}_p^N, || \cdot ||_p) \) is totally disconnected, i.e. the only connected subsets of \( \mathbb{Q}_p^N \) are the empty set and the points. A subset of \( \mathbb{Q}_p^N \) is compact if and only if it is closed and bounded in \( \mathbb{Q}_p^N \), see e.g. [434, Section 1.3], or [18, Section 1.8]. The balls and spheres are compact subsets. Thus \( (\mathbb{Q}_p^N, || \cdot ||_p) \) is a locally compact topological space.
1.3 The Bruhat–Schwartz Space and the Fourier Transform

We will use $\Omega(p^{-r}\|x-a\|_p)$ to denote the characteristic function of the ball $B^n_r(a)$. We will use the notation $1_A$ for the characteristic function of a set $A$.

1.3 The Bruhat–Schwartz Space and the Fourier Transform

A complex-valued function $\varphi$ defined on $\mathbb{Q}_p^N$ is called locally constant if for any $x \in \mathbb{Q}_p^N$ there exists an integer $r(x) \in \mathbb{Z}$ such that

$$\varphi(x + x') = \varphi(x) \quad \text{for all} \ x' \in B^n_r(x). \quad (1.1)$$

The $\mathbb{C}$-vector space of locally constant functions will be denoted as $\mathcal{E}(\mathbb{Q}_p^N)$.

A function $\varphi: \mathbb{Q}_p^N \to \mathbb{C}$ is called a Bruhat–Schwartz function (or a test function) if it is locally constant with compact support. Any test function can be represented as a linear combination, with complex coefficients, of characteristic functions of balls.

The $\mathbb{C}$-vector space of Bruhat–Schwartz functions is denoted by $\mathcal{D}(\mathbb{Q}_p^N) := \mathcal{D}$. For $\varphi \in \mathcal{D}(\mathbb{Q}_p^N)$, the largest number $l = l(\varphi)$ satisfying (1.1) is called the exponent of local constancy (or the parameter of constancy) of $\varphi$. We also say that $p^l$ is the diameter of constancy of $\varphi$. We denote by $\mathcal{D}_l(\mathbb{Q}_p^N)$ the $\mathbb{R}$-vector space of real-valued test functions.

We denote by $\mathcal{D}_M(\mathbb{Q}_p^N) := \mathcal{D}_M^l$, the finite-dimensional space of test functions from $\mathcal{D}(\mathbb{Q}_p^N)$ having supports in the ball $B^n_k$ and with parameters of local constancy $\geq l$.

The following embeddings hold: $\mathcal{D}_M^l \subset \mathcal{D}_M^l'$ for $M \leq M'$, $l' \leq l$.

Any function $\varphi \in \mathcal{D}_M^l$ can be represented in the following way:

$$\varphi(x) = \sum_{v=1}^{p^{N(M-l)}} \varphi(a^v)\Omega(p^{-l}\|x-a^v\|_p), \quad x \in \mathbb{Q}_p^N, \ \varphi \in \mathcal{D}(\mathbb{Q}_p^N), \quad (1.2)$$

where $\Omega(p^{-l}\|x-a^v\|_p)$ is a characteristic function of ball $B^n_k(a^v)$, and the points $a^v = (a^v_1, \ldots, a^v_{N-l}) \in B_{N-l}^k$ do not depend on $\varphi$ and are such that the balls $B^n_{2k}(a^v)$, $v = 1, \ldots, p^{N(M-l)}$ are disjoint and cover $B_{N-l}^k$, see e.g. [18, Lemma 4.3.1].

Convergence in $\mathcal{D}(\mathbb{Q}_p^N)$ is defined in the following way: $\varphi_k \to 0$, $k \to \infty$, in $\mathcal{D}(\mathbb{Q}_p^N)$ if and only if (i) $\varphi_k \subset B^n_M$ with $M$ and $l$ independent of $k$; (ii) $\varphi_k$ uniformly $0$ in $\mathbb{Q}_p^N$.

Set $\mathcal{D}_M(\mathbb{Q}_p^N) := \mathcal{D}_M = \lim_{l \to \infty} \mathcal{D}_M^l$. Then $\mathcal{D}(\mathbb{Q}_p^N) = \lim_{M \to \infty} \mathcal{D}_M$. The space $\mathcal{D}(\mathbb{Q}_p^N)$ is a complete locally convex topological vector space. If $U$ is an open subset of $\mathbb{Q}_p^N$, $\mathcal{D}(U)$ denotes the space of test functions with supports contained in $U$, then $\mathcal{D}(U)$ is dense in

$$L^p(U) = \left\{ \varphi: U \to \mathbb{C}; \|\varphi\|_p = \left\{ \int |\varphi(x)|^p d^N x \right\}^{1/p} < \infty \right\},$$

where $d^N x$ is the Haar measure on $\mathbb{Q}_p^N$ normalized by the condition $\text{vol}(B^n_k) = 1$, for $1 \leq \rho < \infty$, see e.g. [18, Section 4.3]. We will also use the simplified notation $L^p$, $1 \leq \rho < \infty$, if there is no danger of confusion. We will denote by $\mathcal{D}_R(U)$, the $\mathbb{R}$-space...
of test functions with support in $U$, and by $L^p_\mathcal{E}(U)$, $1 \leq \rho < \infty$, the real counterpart of $L^p(U)$, $1 \leq \rho < \infty$.

1.3.1 The Fourier Transform of Test Functions

Given $\xi = (\xi_1, \ldots, \xi_N)$ and $y = (y_1, \ldots, y_N) \in \mathbb{Q}_p^N$, we set $x \cdot y := \sum_{j=1}^N \xi_j y_j$. The Fourier transform of $\varphi \in \mathcal{D}(\mathbb{Q}_p^N)$ is defined as

$$(\mathcal{F}\varphi)(\xi) = \int_{\mathbb{Q}_p^N} \chi_p(x) \varphi(x) d^N x,$$

where $d^N x$ is the normalized Haar measure on $\mathbb{Q}_p^N$. The Fourier transform is a linear isomorphism from $\mathcal{D}(\mathbb{Q}_p^N)$ onto itself satisfying $(\mathcal{F}(\mathcal{F}\varphi))(\xi) = \varphi(-\xi)$, see e.g. [18, Section 4.8]. More precisely, if $\varphi \in \mathcal{D}'_M$ then $\hat{\varphi} \in \mathcal{D}''_{-M}$. We will also use the notation $\mathcal{F}x \rightarrow \xi$ and $\hat{\varphi}$ for the Fourier transform of $\varphi$.

In the definition of the Fourier transform, the bilinear form $x \cdot y$ can be replaced for any symmetric non-degenerate bilinear form $\mathcal{B}(\xi, x)$. We will use such Fourier transforms in Chapter 11. In Chapter 3, we will also use the symbols $\mathcal{F}, \hat{\varphi}$ to denote the Fourier transform in $\mathbb{R}^n$.

1.4 Distributions

Let $\mathcal{D}'(\mathbb{Q}_p^N)$ denote the $\mathbb{C}$-vector space of all continuous functionals (distributions) on $\mathcal{D}(\mathbb{Q}_p^N)$. We endow $\mathcal{D}'(\mathbb{Q}_p^N)$ with weak topology. We denote by $\mathcal{D}'_c(\mathbb{Q}_p^N)$ the real analog of $\mathcal{D}'(\mathbb{Q}_p^N)$. The natural pairing $\mathcal{D}'(\mathbb{Q}_p^N)$ and $\mathcal{D}(\mathbb{Q}_p^N)$ is denoted as $(T, \varphi)$ for $T \in \mathcal{D}'(\mathbb{Q}_p^N)$ and $\varphi \in \mathcal{D}(\mathbb{Q}_p^N)$. Convergence in $\mathcal{D}'(\mathbb{Q}_p^N)$ is defined as the weak convergence $T_k \rightarrow 0, k \rightarrow \infty$, in $\mathcal{D}'(\mathbb{Q}_p^N)$ if $(T_k, \varphi) \rightarrow 0, k \rightarrow \infty$, for all $\varphi \in \mathcal{D}(\mathbb{Q}_p^N)$. The space $\mathcal{D}'(\mathbb{Q}_p^N)$ agrees with the algebraic dual of $\mathcal{D}(\mathbb{Q}_p^N)$, i.e. all functionals on $\mathcal{D}(\mathbb{Q}_p^N)$ are continuous. In addition, $\mathcal{D}'(\mathbb{Q}_p^N)$ is complete, i.e. if $T_k \rightarrow T$, $k \rightarrow \infty$, then there exists a functional $T \in \mathcal{D}'(\mathbb{Q}_p^N)$ such that $T_k \rightarrow T, k \rightarrow \infty$ in $\mathcal{D}'(\mathbb{Q}_p^N)$, see e.g. [18, Section 4.4].

Let $U$ be an open subset of $\mathbb{Q}_p^N$. A distribution $T \in \mathcal{D}'(U)$ vanishes on $V \subset U$ if $(T, \varphi) = 0$ for all $\varphi \in \mathcal{D}(V)$. Let $U_T \subset U$ be the maximal open subset on which the distribution $T$ vanishes. The support of $T$ is the complement of $U_T$ in $U$. We denote it by supp $T$.

Given a fixed test function $\theta$ and a distribution $T \in \mathcal{D}'(\mathbb{Q}_p^N)$, we define the distribution $\partial T$ by the formula $(\partial T, \varphi) = (T, \partial \varphi)$ for $\varphi \in \mathcal{D}(\mathbb{Q}_p^N)$. We say that a distribution $T \in \mathcal{D}'(\mathbb{Q}_p^N)$ has compact support if there exists a $k \in \mathbb{Z}$ such that $\Delta_k T = 0$ in $\mathcal{D}'(\mathbb{Q}_p^N)$, where $\Delta_k(x) := \Omega(p^{-k} \|x\|_p)$.

Every $f \in \mathcal{E}(\mathbb{Q}_p^N)$, or more generally in $L^1_{\text{loc}}$, defines a distribution $f \in \mathcal{D}'(\mathbb{Q}_p^N)$ by the formula

$$(f, \varphi) = \int_{\mathbb{Q}_p^N} f(x) \varphi(x) d^N x.$$

Such distributions are called regular distributions.
1.4 Distributions

1.4.1 The Fourier Transform of a Distribution

The Fourier transform \( \mathcal{F}[T] \) of a distribution \( T \in \mathcal{D}'(\mathbb{Q}_p^N) \) is defined by

\[
(\mathcal{F}[T], \varphi) = (T, \mathcal{F}[\varphi]) \quad \text{for all } \varphi \in \mathcal{D}(\mathbb{Q}_p^N).
\]

The Fourier transform \( T \rightarrow \mathcal{F}[T] \) is a linear (and continuous) isomorphism from \( \mathcal{D}'(\mathbb{Q}_p^N) \) onto \( \mathcal{D}'(\mathbb{Q}_p^N) \). Furthermore, \( T = \mathcal{F}[\mathcal{F}[T](-\xi)] \).

Let \( A \) be a matrix, \( \det A \neq 0 \) and \( b \in \mathbb{Q}_p^N \). Then for a distribution \( T \in \mathcal{D}'(\mathbb{Q}_p^N) \) one has

\[
\mathcal{F}[T(Ax + b)](\xi) = |\det A|^{-1}_p \chi_p(-A^{-1}b \cdot \xi) \mathcal{F}[T(x)]((A^*)^{-1}\xi),
\]

where \( A^* \) is the transpose matrix.

Let \( T \in \mathcal{D}'(\mathbb{Q}_p^N) \) be a distribution. Then \( \operatorname{supp} T \subset B_N^N \) if and only if \( \mathcal{F}[T] \in \tilde{E}(\mathbb{Q}_p^N) \), where the exponent of local constancy of \( \mathcal{F}[T] \) is \( \geq -N \). In addition

\[
\mathcal{F}[T](\xi) = (T(y), \Delta_N(y)\chi_p(\xi \cdot y)),
\]

see e.g. [18, Section 4.9].

1.4.2 The Direct Product of Distributions

Given \( F \in \mathcal{D}'(\mathbb{Q}_p^N) \) and \( G \in \mathcal{D}'(\mathbb{Q}_m^N) \), their direct product \( F \times G \) is defined by the formula

\[
(F(x) \times G(y), \varphi(x, y)) = (F(x), (G(y), \varphi(x, y))) \quad \text{for } \varphi(x, y) \in \mathcal{D}(\mathbb{Q}_{p,m}^{N+N}).
\]

The direct product is commutative: \( F \times G = G \times F \). In addition the direct product is continuous with respect to the joint factors.

1.4.3 The Convolution of Distributions

Given \( F, G \in \mathcal{D}'(\mathbb{Q}_p^N) \), their convolution \( F \ast G \) is defined by

\[
(F \ast G, \varphi) = \lim_{k \to \infty} (F(y) \times G(x), \Delta_k(x)\varphi(x + y))
\]

if the limit exists for all \( \varphi \in \mathcal{D}(\mathbb{Q}_p^N) \). We recall that, if \( F \ast G \) exists, then \( G \ast F \) exists and \( F \ast G = G \ast F \), see e.g. [434, Section 7.1]. If \( F, G \in \mathcal{D}'(\mathbb{Q}_p^N) \) and \( \supp G \subset B_N^N \), then the convolution \( F \ast G \) exists, and it is given by the formula

\[
(F \ast G, \varphi) = (F(y) \times G(x), \Delta_N(x)\varphi(x + y)) \quad \text{for } \varphi \in \mathcal{D}(\mathbb{Q}_p^N).
\]

In the case in which \( G = \psi \in \mathcal{D}(\mathbb{Q}_p^N) \), \( F \ast \psi \) is a locally constant function given by

\[
(F \ast \psi)(y) = (F(x), \psi(y - x)),
\]

see e.g. [434, Section 7.1].
1.4.4 The Multiplication of Distributions

Set $\delta_k(x) := p^{N_k} \Omega(p^k \|x\|_p)$ for $k \in \mathbb{N}$. Given $F, G \in D'(\mathbb{Q}_p^N)$, their product $F \cdot G$ is defined by

$$(F \cdot G, \varphi) = \lim_{k \to \infty} (G, (F * \delta_k)\varphi)$$

if the limit exists for all $\varphi \in D(\mathbb{Q}_p^N)$. If the product $F \cdot G$ exists then the product $G \cdot F$ exists and they are equal.

We recall that the existence of the product $F \cdot G$ is equivalent to the existence of $\mathcal{F}[F] * \mathcal{F}[G]$. In addition, $\mathcal{F}[F \cdot G] = \mathcal{F}[F] * \mathcal{F}[G]$ and $\mathcal{F}[F * G] = \mathcal{F}[F] \cdot \mathcal{F}[G]$, see e.g. [434, Section 7.5].

1.5 Some Function Spaces

1.5.1 $p$-Adic Lizorkin Spaces

In papers [314], [315] the (real) Lizorkin spaces invariant with respect to action of real fractional operators were introduced. By [15], [16], the $p$-adic Lizorkin space of test functions is defined as the space $\Phi(\mathbb{Q}_p^N) \subset D(\mathbb{Q}_p^N)$ of mean-zero test functions. Topology on this set is defined by restriction of the topology on $D(\mathbb{Q}_p^N)$. Equivalently the space $\Phi(\mathbb{Q}_p^N)$ can be defined as the Fourier image of the space $\Psi(\mathbb{Q}_p^N)$ of test functions equal to zero in zero.

Let us denote by $\Phi'(\mathbb{Q}_p^N)$ and $\Psi'(\mathbb{Q}_p^N)$ respectively the spaces topologically dual to $\Phi(\mathbb{Q}_p^N)$ and $\Psi(\mathbb{Q}_p^N)$. We call $\Phi(\mathbb{Q}_p^N)$ the Lizorkin space of $p$-adic distributions. Let $\Psi^-(\mathbb{Q}_p^N)$ and $\Phi^-(\mathbb{Q}_p^N)$ respectively be the subspaces of functionals in $D'(\mathbb{Q}_p^N)$ orthogonal to $\Psi(\mathbb{Q}_p^N)$ and $\Phi(\mathbb{Q}_p^N)$, i.e. $\Psi^-(\mathbb{Q}_p^N) = \{T \in D'(\mathbb{Q}_p^N) : T = C\delta, C \in \mathbb{C}\}$ and $\Phi^-(\mathbb{Q}_p^N) = \{T \in D'(\mathbb{Q}_p^N) : T = C, C \in \mathbb{C}\}$. Then

$$\Phi'(\mathbb{Q}_p^N) = D'(\mathbb{Q}_p^N)/\Phi^-(\mathbb{Q}_p^N), \quad \Psi'(\mathbb{Q}_p^N) = D'(\mathbb{Q}_p^N)/\Psi^-(\mathbb{Q}_p^N),$$

(1.4)

cf. [15].

Therefore the space $\Phi'(\mathbb{Q}_p^N)$ is obtained from $D'(\mathbb{Q}_p^N)$ by factorization over constants (two distributions in $D'(\mathbb{Q}_p^N)$ which differ by a constant are equal as elements of $\Phi'(\mathbb{Q}_p^N)$).

The Fourier transform of distributions $F \in \Phi'(\mathbb{Q}_p^N)$ and $G \in \Psi'(\mathbb{Q}_p^N)$ is given by the formulae $\mathcal{F}[F]\psi = (F, \mathcal{F}[\psi])$, for all $\psi \in \Psi'(\mathbb{Q}_p^N)$, and $\mathcal{F}[G]\phi = (G, \mathcal{F}[\phi])$, for all $\phi \in \Phi'(\mathbb{Q}_p^N)$. One can see that $\mathcal{F}[\Phi'(\mathbb{Q}_p^N)] = \Psi'(\mathbb{Q}_p^N)$ and $\mathcal{F}[\Psi'(\mathbb{Q}_p^N)] = \Phi'(\mathbb{Q}_p^N)$, [15].

The Vladimirov operator is defined by the formula

$$(D^\alpha \varphi)(x) = \mathcal{F}^{-1}(\|\xi\|_p^\alpha \mathcal{F}_{\mathcal{S}_d} \varphi).$$
1.5 Some Function Spaces

This operator maps the space $\Phi(Q^N_p)$ into itself. For $\alpha > 0$ (in the one-dimensional case) there is the integral representation

$$D^\alpha \varphi(x) = \frac{1}{\Gamma_p(-\alpha)} \int_{Q_p} \frac{\varphi(x) - \varphi(y)}{|x-y|_p^{1+\alpha}} dy,$$

(1.5)

where $\Gamma_p(-\alpha) = (p^\alpha - 1)/(1 - p^{-1-\alpha})$ is the $p$-adic $\Gamma$-function, cf. [434, Section IX, (1.1)].