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Andrei Yu. Khrennikov , Sergei V. Kozyrev , W. A. Zúñiga-Galindo
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p-Adic Analysis: Essential Ideas and Results

In this chapter, we present, without proofs, the essential aspects of, and basic results on, p-adic functional analysis needed in the book. For a detailed exposition on p-adic analysis the reader may consult [18], [402], [434].

1.1 The Field of p-Adic Numbers

Throughout this book p will denote a prime number. The field of p-adic numbers $\mathbb{Q}_p$ is defined as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the p-adic norm $|\cdot|_p$, which is defined as

$$|x|_p = \begin{cases} 0 & \text{if } x = 0 \\ p^{-\gamma} & \text{if } x = p^{\gamma}\dfrac{a}{b}, \end{cases}$$

where a and b are integers coprime with p. The integer $\gamma := \operatorname{ord}(x)$, with $\operatorname{ord}(0) := +\infty$, is called the *$p$-adic order of x*. We extend the p-adic norm to $\mathbb{Q}_p^N$ by taking

$$||x||_p := \max_{1\le i\le N} |x_i|_p, \quad \text{for } x = (x_1, \dots, x_N) \in \mathbb{Q}_p^N.$$

We define $\operatorname{ord}(x) = \min_{1\le i\le N}\{\operatorname{ord}(x_i)\}$, then $||x||_p = p^{-\operatorname{ord}(x)}$. The norm $||\cdot||_p$ satisfies

$$\|x + y\|_p \le \max(\|x\|_p, \|y\|_p),$$

the strong triangle inequality. The metric space $(\mathbb{Q}_p^N, ||\cdot||_p)$ is a complete ultrametric space. As a topological space $\mathbb{Q}_p$ is homeomorphic to a Cantor-like subset of the real line, see e.g. [18], [434]. In Chapter 2 only, we will work with more general norms, namely, norms of type

$$\mathcal{N}_{q_1,\dots,q_N}(x) = \max_{1\le i\le N} q_i |x_i|_p,$$

where the $q_1, \dots, q_N$ are fixed positive numbers.

Any p-adic number $x \neq 0$ has a unique expansion of the form

$$x = p^{\mathrm{ord}(x)} \sum_{j=0}^{\infty} x_i p^j,$$

where $x_j \in \{0, 1, 2, \dots, p-1\}$ and $x_0 \neq 0$. By using this expansion, we define the *fractional part of* $x \in \mathbb{Q}_p$, denoted $\{x\}_p$, as the rational number

$$\{x\}_p = \begin{cases} 0 & \text{if } x = 0 \text{ or } \mathrm{ord}(x) \geq 0 \\ p^{\mathrm{ord}(x)} \sum_{j=0}^{-\mathrm{ord}(x)-1} x_j p^j & \text{if } \mathrm{ord}(x) < 0. \end{cases}$$

In addition, any p-adic number $x \neq 0$ can be represented uniquely as $x = p^{\mathrm{ord}(x)}\mathrm{ac}(x)$ with $|\mathrm{ac}(x)|_p = 1$; $\mathrm{ac}(x)$ is called the *angular component* of x.

1.1.1 Additive Characters

Set $\chi_p(y) := \exp(2\pi i \{y\}_p)$ for $y \in \mathbb{Q}_p$. The map $\chi_p(\cdot)$ is an additive character on $\mathbb{Q}_p$, i.e. a continuous map from $(\mathbb{Q}_p, +)$ into S (the unit circle considered as a multiplicative group) satisfying $\chi_p(x_0 + x_1) = \chi_p(x_0)\chi_p(x_1)$, $x_0, x_1 \in \mathbb{Q}_p$. We notice that χ_p satisfies the following relations:

$$\chi_p(0) = 1, \qquad \chi_p(-x) = \overline{\chi_p(x)} = \chi_p^{-1}(x),$$
$$\chi_p(mx) = \chi_p^m(x), \;\; m \in \mathbb{Z}.$$

The additive characters of $\mathbb{Q}_p$ form an Abelian group which is isomorphic to $(\mathbb{Q}_p, +)$; the isomorphism is given by $\xi \to \chi_p(\xi x)$, see e.g. [18, Section 2.3]. We will call $\chi_p(\cdot)$ the *standard additive character* of $\mathbb{Q}_p$.

1.2 Topology of $\mathbb{Q}_p^N$

For $r \in \mathbb{Z}$, denote by $B_r^N(a) = \{x \in \mathbb{Q}_p^N; ||x - a||_p \leq p^r\}$ the *ball of radius* p^r *with center at* $a = (a_1, \dots, a_N) \in \mathbb{Q}_p^N$, and take $B_r^N(0) := B_r^N$. Note that $B_r^N(a) = B_r(a_1) \times \cdots \times B_r(a_N)$, where $B_r(a_i) := \{x \in \mathbb{Q}_p; |x_i - a_i|_p \leq p^r\}$ is the one-dimensional ball of radius p^r with center at $a_i \in \mathbb{Q}_p$. The ball B_0^N equals the product of N copies of $B_0 = \mathbb{Z}_p$, the *ring of* p*-adic integers*. We also denote by $S_r^N(a) = \{x \in \mathbb{Q}_p^N; ||x - a||_p = p^r\}$ the *sphere of radius* p^r *with center at* $a = (a_1, \dots, a_N) \in \mathbb{Q}_p^N$, and take $S_r^N(0) := S_r^N$. We notice that $S_0^1 = \mathbb{Z}_p^\times$ (the group of units of $\mathbb{Z}_p$), but $(\mathbb{Z}_p^\times)^N \subsetneq S_0^N$. The balls and spheres are both open and closed subsets in $\mathbb{Q}_p^N$. In addition, two balls in $\mathbb{Q}_p^N$ are either disjoint or one is contained in the other.

As a topological space $(\mathbb{Q}_p^N, || \cdot ||_p)$ is totally disconnected, i.e. the only connected subsets of $\mathbb{Q}_p^N$ are the empty set and the points. A subset of $\mathbb{Q}_p^N$ is compact if and only if it is closed and bounded in $\mathbb{Q}_p^N$, see e.g. [434, Section 1.3], or [18, Section 1.8]. The balls and spheres are compact subsets. Thus $(\mathbb{Q}_p^N, || \cdot ||_p)$ is a locally compact topological space.

We will use $\Omega(p^{-r}||x-a||_p)$ to denote the characteristic function of the ball $B_r^N(a)$. We will use the notation 1_A for the characteristic function of a set A.

1.3 The Bruhat–Schwartz Space and the Fourier Transform

A complex-valued function φ defined on $\mathbb{Q}_p^N$ is called *locally constant* if for any $x \in \mathbb{Q}_p^N$ there exists an integer $l(x) \in \mathbb{Z}$ such that

$$\varphi(x+x') = \varphi(x) \quad \text{for all } x' \in B_{l(x)}^N. \tag{1.1}$$

The $\mathbb{C}$-vector space of locally constant functions will be denoted as $\mathcal{E}(\mathbb{Q}_p^N)$.

A function $\varphi\colon \mathbb{Q}_p^N \to \mathbb{C}$ is called a *Bruhat–Schwartz function* (or a *test function*) if it is locally constant with compact support. Any test function can be represented as a linear combination, with complex coefficients, of characteristic functions of balls. The $\mathbb{C}$-vector space of Bruhat–Schwartz functions is denoted by $\mathcal{D}(\mathbb{Q}_p^N) := \mathcal{D}$. For $\varphi \in \mathcal{D}(\mathbb{Q}_p^N)$, the largest number $l = l(\varphi)$ satisfying (1.1) is called the *exponent of local constancy* (or the *parameter of constancy*) *of* φ. We also say that p^l is the *diameter of constancy of* φ. We denote by $\mathcal{D}_{\mathbb{R}}(\mathbb{Q}_p^N)$ the $\mathbb{R}$-vector space of real-valued test functions.

We denote by $\mathcal{D}_M^l(\mathbb{Q}_p^N) := \mathcal{D}_M^l$ the finite-dimensional space of test functions from $\mathcal{D}(\mathbb{Q}_p^N)$ having supports in the ball B_M^N and with parameters of local constancy $\geq l$. The following embeddings hold: $\mathcal{D}_M^l \subset \mathcal{D}_{M'}^{l'}$ for $M \leq M'$, $l' \leq l$.

Any function $\varphi \in \mathcal{D}_M^l$ can be represented in the following way:

$$\varphi(x) = \sum_{\nu=1}^{p^{N(M-l)}} \varphi(a^\nu)\Omega(p^{-l}\|x-a^\nu\|_p), \quad x \in \mathbb{Q}_p^N, \quad \varphi \in \mathcal{D}\big(\mathbb{Q}_p^N\big), \tag{1.2}$$

where $\Omega(p^{-l}\|x-a^\nu\|_p)$ is a characteristic function of ball $B_l^N(a^\nu)$, and the points $a^\nu = (a_1^\nu, \ldots, a_N^\nu) \in B_M^N$ do not depend on φ and are such that the balls $B_l^N(a^\nu)$, $\nu = 1, \ldots, p^{N(M-l)}$ are disjoint and cover B_M^N, see e.g. [18, Lemma 4.3.1].

Convergence in $\mathcal{D}(\mathbb{Q}_p^N)$ is defined in the following way: $\varphi_k \to 0$, $k \to \infty$, in $\mathcal{D}(\mathbb{Q}_p^N)$ if and only if (i) $\varphi_k \subset B_M^l$ with M and l independent of k; (ii) $\varphi_k \xrightarrow{\text{uniformly}} 0$ in $\mathbb{Q}_p^N$.

Set $\mathcal{D}_M(\mathbb{Q}_p^N) := \mathcal{D}_M = \lim \operatorname{ind}_{l\to-\infty}\mathcal{D}_M^l$. Then $\mathcal{D}(\mathbb{Q}_p^N) = \lim\ \operatorname{ind}_{M\to\infty}\mathcal{D}_M$. The space $\mathcal{D}(\mathbb{Q}_p^N)$ is a complete locally convex topological vector space. If U is an open subset of $\mathbb{Q}_p^N$, $\mathcal{D}(U)$ denotes the space of test functions with supports contained in U, then $\mathcal{D}(U)$ is dense in

$$L^\rho(U) = \left\{\varphi : U \to \mathbb{C};\ \|\varphi\|_\rho = \left\{\int |\varphi(x)|^\rho d^N x\right\}^{\frac{1}{\rho}} < \infty\right\},$$

where $d^N x$ is the Haar measure on $\mathbb{Q}_p^N$ normalized by the condition $\mathrm{vol}(B_0^N) = 1$, for $1 \leq \rho < \infty$, see e.g. [18, Section 4.3]. We will also use the simplified notation L^ρ, $1 \leq \rho < \infty$, if there is no danger of confusion. We will denote by $\mathcal{D}_{\mathbb{R}}(U)$, the $\mathbb{R}$-space of test functions with support in U, and by $L^{\rho}_{\mathbb{R}}(U)$, $1 \le \rho < \infty$, the real counterpart of $L^{\rho}(U)$, $1 \le \rho < \infty$.

1.3.1 The Fourier Transform of Test Functions

Given $\xi = (\xi_1, \ldots, \xi_N)$ and $y = (x_1, \ldots, x_N) \in \mathbb{Q}_p^N$, we set $\xi \cdot x := \sum_{j=1}^{N} \xi_j x_j$. The Fourier transform of $\varphi \in \mathcal{D}(\mathbb{Q}_p^N)$ is defined as

$$(\mathcal{F}\varphi)(\xi) = \int_{\mathbb{Q}_p^N} \chi_p(\xi \cdot x)\varphi(x)d^N x \quad \text{for } \xi \in \mathbb{Q}_p^N,$$

where $d^N x$ is the normalized Haar measure on $\mathbb{Q}_p^N$. The Fourier transform is a linear isomorphism from $\mathcal{D}(\mathbb{Q}_p^N)$ onto itself satisfying $(\mathcal{F}(\mathcal{F}\varphi))(\xi) = \varphi(-\xi)$, see e.g. [18, Section 4.8]. More precisely, if $\varphi \in \mathcal{D}_M^l$ then $\widehat{\varphi} \in \mathcal{D}_{-l}^{-M}$. We will also use the notation $\mathcal{F}_{x \to \xi}\varphi$ and $\widehat{\varphi}$ for the Fourier transform of φ.

In the definition of the Fourier transform, the bilinear form $\xi \cdot x$ can be replaced for any symmetric non-degenerate bilinear form $\mathfrak{B}(\xi, x)$. We will use such Fourier transforms in Chapter 11. In Chapter 3, we will also use the symbols $\mathcal{F}, \widehat{\cdot}$ to denote the Fourier transform in $\mathbb{R}^n$.

1.4 Distributions

Let $\mathcal{D}'(\mathbb{Q}_p^N)$ denote the $\mathbb{C}$-vector space of all continuous functionals (distributions) on $\mathcal{D}(\mathbb{Q}_p^N)$. We endow $\mathcal{D}'(\mathbb{Q}_p^N)$ with weak topology. We denote by $\mathcal{D}'_R(\mathbb{Q}_p^N)$ the real analog of $\mathcal{D}'(\mathbb{Q}_p^N)$. The natural pairing $\mathcal{D}'(\mathbb{Q}_p^N) \times \mathcal{D}(\mathbb{Q}_p^N) \to \mathbb{C}$ is denoted as (T, φ) for $T \in \mathcal{D}'(\mathbb{Q}_p^N)$ and $\varphi \in \mathcal{D}(\mathbb{Q}_p^N)$. Convergence in $\mathcal{D}'(\mathbb{Q}_p^N)$ is defined as the weak convergence $T_k \to 0$, $k \to \infty$, in $\mathcal{D}'(\mathbb{Q}_p^N)$ if $(T_k, \varphi) \to 0$, $k \to \infty$, for all $\varphi \in \mathcal{D}(\mathbb{Q}_p^N)$. The space $\mathcal{D}'(\mathbb{Q}_p^N)$ agrees with the algebraic dual of $\mathcal{D}(\mathbb{Q}_p^N)$, i.e. all functionals on $\mathcal{D}(\mathbb{Q}_p^N)$ are continuous. In addition, $\mathcal{D}'(\mathbb{Q}_p^N)$ is complete, i.e. if $T_k - T_j \to 0$, $k, j \to \infty$, then there exists a functional $T \in \mathcal{D}'(\mathbb{Q}_p^N)$ such that $T_k - T \to 0$, $k \to \infty$ in $\mathcal{D}'(\mathbb{Q}_p^N)$, see e.g. [18, Section 4.4].

Let U be an open subset of $\mathbb{Q}_p^N$. A distribution $T \in \mathcal{D}'(U)$ vanishes on $V \subset U$ if $(T, \varphi) = 0$ for all $\varphi \in \mathcal{D}(V)$. Let $U_T \subset U$ be the maximal open subset on which the distribution T vanishes. The support of T is the complement of U_T in U. We denote it by supp T.

Given a fixed test function θ and a distribution $T \in \mathcal{D}'(\mathbb{Q}_p^N)$, we define the distribution θT by the formula $(\theta T, \varphi) = (T, \theta\varphi)$ for $\varphi \in \mathcal{D}(\mathbb{Q}_p^N)$. We say that a distribution $T \in \mathcal{D}'(\mathbb{Q}_p^N)$ has *compact support* if there exists a $k \in \mathbb{Z}$ such that $\Delta_k T = T$ in $\mathcal{D}'(\mathbb{Q}_p^N)$, where $\Delta_k(x) := \Omega(p^{-k}\|x\|_p)$.

Every $f \in \mathcal{E}(\mathbb{Q}_p^N)$, or more generally in L^1_{loc}, defines a distribution $f \in \mathcal{D}'(\mathbb{Q}_p^N)$ by the formula

$$(f, \varphi) = \int_{\mathbb{Q}_p^N} f(x)\varphi(x)d^N x.$$

Such distributions are called *regular distributions*.

1.4.1 The Fourier Transform of a Distribution

The Fourier transform $\mathcal{F}[T]$ of a distribution $T \in \mathcal{D}'(\mathbb{Q}_p^N)$ is defined by

$$(\mathcal{F}[T], \varphi) = (T, \mathcal{F}[\varphi]) \quad \text{for all } \varphi \in \mathcal{D}\big(\mathbb{Q}_p^N\big).$$

The Fourier transform $T \to \mathcal{F}[T]$ is a linear (and continuous) isomorphism from $\mathcal{D}'(\mathbb{Q}_p^N)$ onto $\mathcal{D}'(\mathbb{Q}_p^N)$. Furthermore, $T = \mathcal{F}[\mathcal{F}[T](-\xi)]$.

Let A be a matrix, $\det A \neq 0$ and $b \in \mathbb{Q}_p^N$. Then for a distribution $T \in \mathcal{D}'(\mathbb{Q}_p^N)$ one has

$$\mathcal{F}[T(Ax+b)](\xi) = |\det A|_p^{-1} \chi_p\big(-A^{-1}b \cdot \xi\big)\mathcal{F}[T(x)]\big((A^*)^{-1}\xi\big), \tag{1.3}$$

where A^* is the transpose matrix.

Let $T \in \mathcal{D}'(\mathbb{Q}_p^N)$ be a distribution. Then $\operatorname{supp} T \subset B_N^N$ if and only if $\mathcal{F}[T] \in \widetilde{\mathcal{E}}(\mathbb{Q}_p^N)$, where the exponent of local constancy of $\mathcal{F}[T]$ is $\geq -N$. In addition

$$\mathcal{F}[T](\xi) = (T(y), \Delta_N(y)\chi_p(\xi \cdot y)),$$

see e.g. [18, Section 4.9].

1.4.2 The Direct Product of Distributions

Given $F \in \mathcal{D}'(\mathbb{Q}_p^N)$ and $G \in \mathcal{D}'(\mathbb{Q}_p^m)$, their *direct product* $F \times G$ is defined by the formula

$$(F(x) \times G(y), \varphi(x,y)) = (F(x), (G(y), \varphi(x,y))) \quad \text{for } \varphi(x,y) \in \mathcal{D}\big(\mathbb{Q}_p^{N+m}\big).$$

The direct product is commutative: $F \times G = G \times F$. In addition the direct product is continuous with respect to the joint factors.

1.4.3 The Convolution of Distributions

Given $F, G \in \mathcal{D}'(\mathbb{Q}_p^N)$, their convolution $F * G$ is defined by

$$(F * G, \varphi) = \lim_{k\to\infty} (F(y) \times G(x), \Delta_k(x)\varphi(x+y))$$

if the limit exists for all $\varphi \in \mathcal{D}(\mathbb{Q}_p^N)$. We recall that, if $F * G$ exists, then $G * F$ exists and $F * G = G * F$, see e.g. [434, Section 7.1]. If $F, G \in \mathcal{D}'(\mathbb{Q}_p^N)$ and $\operatorname{supp} G \subset B_N^N$, then the convolution $F * G$ exists, and it is given by the formula

$$(F * G, \varphi) = (F(y) \times G(x), \Delta_N(x)\varphi(x+y)) \quad \text{for } \varphi \in \mathcal{D}\big(\mathbb{Q}_p^N\big).$$

In the case in which $G = \psi \in \mathcal{D}(\mathbb{Q}_p^N)$, $F * \psi$ is a locally constant function given by

$$(F * \psi)(y) = (F(x), \psi(y-x)),$$

see e.g. [434, Section 7.1].

1.4.4 The Multiplication of Distributions

Set $\delta_k(x) := p^{Nk}\Omega(p^k\|x\|_p)$ for $k \in \mathbb{N}$. Given $F, G \in \mathcal{D}'(\mathbb{Q}_p^N)$, their product $F \cdot G$ is defined by

$$(F \cdot G, \varphi) = \lim_{k\to\infty} (G, (F * \delta_k)\varphi)$$

if the limit exists for all $\varphi \in \mathcal{D}(\mathbb{Q}_p^N)$. If the product $F \cdot G$ exists then the product $G \cdot F$ exists and they are equal.

We recall that the existence of the product $F \cdot G$ is equivalent to the existence of $\mathcal{F}[F] * \mathcal{F}[G]$. In addition, $\mathcal{F}[F \cdot G] = \mathcal{F}[F] * \mathcal{F}[G]$ and $\mathcal{F}[F * G] = \mathcal{F}[F] \cdot \mathcal{F}[G]$, see e.g. [434, Section 7.5].

1.5 Some Function Spaces

1.5.1 p-Adic Lizorkin Spaces

In papers [314], [315] the (real) Lizorkin spaces invariant with respect to action of real fractional operators were introduced. By [15], [16], the p-adic Lizorkin space of test functions is defined as the space $\Phi(\mathbb{Q}_p^N) \subset \mathcal{D}(\mathbb{Q}_p^N)$ of mean-zero test functions. Topology on this set is defined by restriction of the topology on $\mathcal{D}(\mathbb{Q}_p^N)$. Equivalently the space $\Phi(\mathbb{Q}_p^N)$ can be defined as the Fourier image of the space $\Psi(\mathbb{Q}_p^N)$ of test functions equal to zero in zero.

Let us denote by $\Phi'(\mathbb{Q}_p^N)$ and $\Psi'(\mathbb{Q}_p^N)$ respectively the spaces topologically dual to $\Phi(\mathbb{Q}_p^N)$ and $\Psi(\mathbb{Q}_p^N)$. We call $\Phi'(\mathbb{Q}_p^N)$ the Lizorkin space of p-adic distributions. Let $\Psi^{\perp}(\mathbb{Q}_p^N)$ and $\Phi^{\perp}(\mathbb{Q}_p^N)$ respectively be the subspaces of functionals in $\mathcal{D}'(\mathbb{Q}_p^N)$ orthogonal to $\Psi(\mathbb{Q}_p^N)$ and $\Phi(\mathbb{Q}_p^N)$, i.e. $\Psi^{\perp}(\mathbb{Q}_p^N) = \{T \in \mathcal{D}'(\mathbb{Q}_p^N) : T = C\delta,\ C \in \mathbb{C}\}$ and $\Phi^{\perp}(\mathbb{Q}_p^N) = \{T \in \mathcal{D}'(\mathbb{Q}_p^N) : T = C,\ C \in \mathbb{C}\}$. Then

$$\Phi'\big(\mathbb{Q}_p^N\big) = \mathcal{D}'\big(\mathbb{Q}_p^N\big)/\Phi^{\perp}\big(\mathbb{Q}_p^N\big), \quad \Psi'\big(\mathbb{Q}_p^N\big) = \mathcal{D}'\big(\mathbb{Q}_p^N\big)/\Psi^{\perp}\big(\mathbb{Q}_p^N\big), \tag{1.4}$$

cf. [15].

Therefore the space $\Phi'(\mathbb{Q}_p^N)$ is obtained from $\mathcal{D}'(\mathbb{Q}_p^N)$ by factorization over constants (two distributions in $\mathcal{D}'(\mathbb{Q}_p^N)$ which differ by a constant are equal as elements of $\Phi'(\mathbb{Q}_p^N)$).

The Fourier transform of distributions $F \in \Phi'(\mathbb{Q}_p^N)$ and $G \in \Psi'(\mathbb{Q}_p^N)$ is given by the formulae $(\mathcal{F}[F], \psi) = (F, \mathcal{F}[\psi])$, for all $\psi \in \Psi(\mathbb{Q}_p^N)$, and $(\mathcal{F}[G], \phi) = (G, \mathcal{F}[\phi])$, for all $\phi \in \Phi(\mathbb{Q}_p^N)$. One can see that $\mathcal{F}[\Phi'(\mathbb{Q}_p^N)] = \Psi'(\mathbb{Q}_p^N)$ and $\mathcal{F}[\Psi'(\mathbb{Q}_p^N)] = \Phi'(\mathbb{Q}_p^N)$, [15].

The Vladimirov operator is defined by the formula

$$(\boldsymbol{D}^{\alpha}\varphi)(x) = \mathcal{F}^{-1}\big(|\xi|_p^{\alpha}\mathcal{F}_{x\to\xi}\varphi\big).$$

This operator maps the space $\Phi(\mathbb{Q}_p^N)$ into itself. For $\alpha > 0$ (in the one-dimensional case) there is the integral representation

$$\boldsymbol{D}^{\alpha}\varphi(x) = \frac{1}{\Gamma_p(-\alpha)} \int_{\mathbb{Q}_p} \frac{\varphi(x) - \varphi(y)}{|x-y|_p^{1+\alpha}}\, dy, \tag{1.5}$$

where $\Gamma_p(-\alpha) = (p^{\alpha} - 1)/(1 - p^{-1-\alpha})$ is the p-adic Γ-function, cf. [434, Section IX, (1.1)].