LECTURES ON LOGARITHMIC ALGEBRAIC GEOMETRY

This graduate textbook offers a self-contained introduction to the concepts and techniques of logarithmic geometry, a key tool for analyzing compactification and degeneration in algebraic geometry and number theory. It features a systematic exposition of the foundations of the field, from the basic results on convex geometry and commutative monoids to the theory of logarithmic schemes and their de Rham and Betti cohomology. The book will be of use to graduate students and researchers working in algebraic, analytic, and arithmetic geometry, as well as related fields.

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Lectures on Logarithmic Algebraic Geometry

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1 Introduction

1.1 Motivation

Logarithmic geometry was developed to deal with two fundamental and related problems in algebraic geometry: compactification and degeneration. A key aspect of algebraic geometry is that it is essentially global in nature. Algebraic varieties can be compactified: any separated scheme S of finite type over a field k admits an open embedding $j\colon S \hookrightarrow T$, with T/k proper and with S Zariski dense in T [55, 9]. Since proper schemes are much easier to study than general schemes, it is often convenient to work with T even if it is the original scheme S that is of primary interest. It then becomes necessary to keep track of the complement $Z := T \setminus S$ and to study how functions, differential forms, sheaves, and other geometric objects on T behave near Z, and to have a mechanism to extract S from T. In differential topology, these problems are often addressed by working with manifolds with boundary, and logarithmic geometry can be thought of as a substitute for, or version of, the notion of "algebraic variety with boundary." Indeed, log schemes over the field of complex numbers have "Betti realizations,"[1] and the Betti realizations of logarithmically smooth log schemes are topological manifolds with boundary.

The compactification problem is related to the phenomenon of degeneration. A scheme S often arises as a moduli space, for example, a space parameterizing smooth proper schemes of a certain type. If S is a fine moduli space, there is a smooth proper morphism $f\colon U \to S$ whose fibers are the objects one wants to classify. One can then hope to find a compactification T of S such that the boundary points parameterize "decorated degenerations" of the original objects. In this case there should be a proper and flat (but not smooth) $g\colon X \to T$ extending $f\colon U \to S$. Then one is left with the problem of comparing f to g and in particular of analyzing the behavior of g near $Y := X \setminus U$. In many cases one can obtain important information about the original family f by studying the degenerate family over Z. A typical example is the compactification of the moduli stack of smooth curves by the moduli stack of stable curves.

The problems of compactification and degeneration are thus manifest in a

[1] Betti realizations of log schemes were introduced by Kato and Nakayama and are often called "Kato–Nakayama spaces."

diagram of the form:

$$\begin{array}{ccccc} U & \hookrightarrow & X & \hookleftarrow & Y \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g_{|_Y}} \\ S & \hookrightarrow & T & \hookleftarrow & Z. \end{array}$$

It turns out that in many such cases there is a natural way to equip X and T with *log structures,* which somehow "remember" U and S and are compatible with g. Then $g\colon X \to T$ becomes a *morphism of log schemes* and inherits many of the nice features of f. The log structures on X and T restrict in a natural way to Y and Z, and the resulting morphism of log schemes $g_{|_Y}\colon Y \to Z$ still remembers useful information about f, thanks to the "decoration" provided by the log structures on Y and Z.

In good cases, the log structures on f, X, and T render the morphism f *logarithmically smooth*, which makes it much easier to study than the underlying morphism of schemes. The concept of smoothness for log schemes fits very naturally into Grothendieck's geometric deformation theory. Furthermore, Betti realizations of proper log smooth morphisms behave in some respects like topological fibrations (see [44] and [57]). The fact that this picture works so well both in topological and in arithmetical settings is one of the main justifications for the theory of log geometry.

Let us illustrate how log geometry works in the most basic case, that of a (possibly partial) compactification. Let $j\colon U \to X$ be an open immersion, with complementary closed immersion $i\colon Y \to X$. Then Y (and hence U) is determined by the sheaf $I_Y \subseteq \mathcal{O}_X$ consisting of those local sections of $\mathcal{O}_X$ whose restriction to Y vanishes, a sheaf of ideals of $\mathcal{O}_X$. However, it is not Y but rather U that is our primary interest, so instead we consider the subsheaf $\mathcal{M}_{U/X}$ of $\mathcal{O}_X$ consisting of the local sections of $\mathcal{O}_X$ whose restriction to U is invertible. If f and g are sections of $\mathcal{M}_{U/X}$, then so is fg, but $f+g$ need not be. Thus $\mathcal{M}_{U/X}$ is not a sheaf of rings, but it is a sheaf of *submonoids* of the multiplicative sheaf of monoids underlying $\mathcal{O}_X$. Note that $\mathcal{M}_{U/X}$ contains the sheaf of units $\mathcal{O}_X^*$, and if X is integral, the quotient $\mathcal{M}_{U/X}/\mathcal{O}_X^*$ can be naturally identified[2] with the sheaf of effective Cartier divisors on X with support in the complement Y of U in X. The morphism of sheaves of monoids $\alpha_{U/X}\colon \mathcal{M}_{U/X} \to \mathcal{O}_X$ (inclusion) is a *logarithmic structure*, called the *compactifying log structure* associated to the embedding j. In good cases this log structure "remembers" the inclusion $U \to X$ and furthermore satisfies a technical *coherence* condition that makes

[2] This identification takes the class of a local section m of $\mathcal{M}_{U/X}$ to the inverse of the (invertible) ideal sheaf generated by $\alpha_{U/X}(m)$.

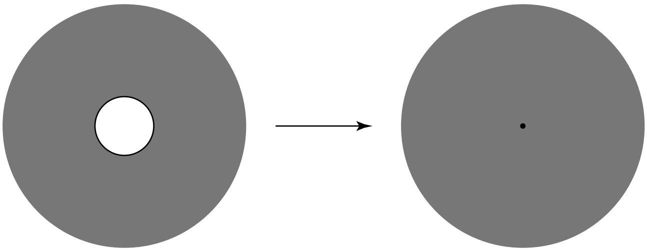

Figure 1.1.1 Compactifying an open immersion

it manageable. In the category of log schemes, the open immersion j fits into a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\tilde{j}} & (X, \alpha_{U/X}) \\ & \searrow^{j} & \downarrow{\scriptstyle \tau_{U/X}} \\ & & X. \end{array}$$

This diagram provides a relative compactification of the open immersion j. The map $\tau_{U/X}$ is proper but the map $\tilde{j}$ somehow preserves much of the essential nature of the original open immersion j: in good cases, it behaves like a local homotopy equivalence. We can imagine that the log structure $\alpha_{U/X}$ cuts away or blows up enough of X to make it look like U, but leaves enough of a boundary for it to remain compact. It is in this sense that the log scheme $(X, \alpha_{U/X})$ plays the role of an "algebraic variety with boundary." For example, in the case of the standard embedding of $\mathbf{G_m} \to \mathbf{A}^1$, the corresponding log scheme $(\mathbf{A}^1, \alpha)$ behaves very much like the complex plane in which the origin is blown up to become a circle, as shown in Figure 1.1.1. The morphism in this picture can be identified with the multiplication map $\mathbf{R}_{\geq} \times \mathbf{S}^1 \to \mathbf{C}$, where $\mathbf{R}_{\geq}$ is the set of nonnegative real numbers and $\mathbf{S}^1$ is the set of complex numbers of absolute value one. This "real blowup" resolves the ambiguity of polar coordinates. It serves as a proper model of the inclusion $\mathbf{G_m} \to \mathbf{A}^1$, whose homotopy theory it closely resembles. These ideas will be made more precise in Section 1 of Chapter V, where we discuss Betti realizations of log schemes. In particular, Theorem V.1.3.1 shows that the Betti realization of a (logarithmically) smooth log scheme over $\mathbf{C}$ really is a topological manifold with boundary.

In general, a *log structure* on a scheme X is a morphism of sheaves of commutative monoids $\alpha\colon \mathcal{M} \to \mathcal{O}_X$ inducing an isomorphism $\alpha^{-1}(\mathcal{O}_X^*) \to \mathcal{O}_X^*$. We do not require α to be injective. In particular, sections of $\mathcal{M}$ can map to zero in $\mathcal{O}_X$, although in good cases $\mathcal{M}$ is *integral,* so that on $\mathcal{M}$, multiplication by any local section is injective. The tension between these behaviors accounts for much of the power, as well as many of the technical difficulties, of log geometry, particularly those involving fiber products. The flexibility and functoriality of log structures allow us to restrict a compactifying log structure $\alpha_{U/X}$ to $X \setminus U$, where sections of the sheaf of monoids $\mathcal{M}$ keep track of the "ghosts of vanishing coordinates."

The naturality of these constructions allows them to work in appropriate relative settings, for example, in the context of semistable reduction. Let X be a regular scheme, let T be the spectrum of a discrete valuation ring, and let $f\colon X \to T$ be a flat and proper morphism whose generic fiber X_τ/τ is smooth and whose special fiber is a reduced divisor with normal crossings. Then the compactifying log structures α_X and α_T associated as above to the open embeddings $X_\tau \to X$ and $\tau \to T$ fit into a morphism of log schemes

$$f\colon (X, \alpha_X) \to (T, \alpha_T),$$

which is in fact logarithmically smooth.

The value of the machinery of log geometry must be judged by its applications to problems outside the theory itself. A detailed discussion of any of these would be beyond the scope of this book, and we can only point readers to the literature. Historically, the first (and perhaps still most striking) such application is in the proof, due to Hyodo and Kato [37], Kato [48], Tsuji [75], Faltings [19] [18], and others, of the "C_{st} conjecture" in p-adic Hodge theory. Indeed, log geometry began as an attempt to discern what additional structure on the special fiber of a semistable reduction was needed to define a "limiting crystalline cohomology," in analogy to Steenbrink's construction of limiting mixed Hodge structures in the complex analytic context [73], [74]. In ℓ-adic cohomology, the main applications have been to the Bloch conductor formula [51] and higher dimensional Ogg–Shafarevich formulas [1] and to results on resolution, purity, and duality [42]. Log geometry has also been notably used in the theory of mirror symmetry [24] and the study of compactifications of moduli spaces of curves [47], [68], abelian varieties [64], K3 surfaces [63], and toric Hilbert schemes [65].

1.2 Roots

The development of logarithmic geometry, like that of any organism, began well before its official birth, and was preceded by many classical methods dealing with the problems of compactification and degeneration. These include most notably the theories of toroidal embeddings, of differential forms and equations with log poles and/or regular singularities, and of vanishing cycles and monodromy. Logarithmic geometry was influenced by all these ideas and provides a language that incorporates and extends them in functorial and systematic ways.

Logarithmic structures fit so naturally with the usual building blocks of schemes that it is possible, and in most (but not all) cases, straightforward and natural, to adapt many of the standard techniques and intuitions of algebraic geometry to the logarithmic context. Log geometry seems to be especially compatible with infinitesimal techniques, including Grothendieck's notions of smoothness, differentials, and differential operators. The sheaf of Kähler differentials of a logarithmic scheme (X, α_X), constructed from Grothendieck's deformation-theoretic viewpoint, coincides with the classical sheaf of differential forms of X with log poles along $X \setminus U$; this fact is one justification for the terminology. Furthermore, any toric variety (with the log structure corresponding to the dense open torus it contains) is log smooth, and the theory of toroidal embeddings is essentially equivalent to the study of (logarithmically) smooth log schemes over a field.

1.3 Goals

Our aim in this book is to provide an introduction to the basic notions and techniques of log geometry that is accessible to graduate students with a basic knowledge of algebraic geometry. We hope the material will also be useful to researchers in other areas of geometry, to which we believe the theory can be profitably adopted, as has already been done in the case of complex analytic geometry. For the sake of concreteness, we work systematically with schemes as locally ringed spaces, although it certainly would have been possible and profitable to develop the theory for complex analytic varieties, or for algebraic spaces or stacks. Even in the case of schemes, it is quite valuable to work locally in the étale topology, and we shall allow ourselves to do so, although we do not use the language of topos theory. (That more powerful approach is taken in the very thorough treatment in [22].)

Just as scheme theory starts with the study of commutative rings, log geometry starts with the study of commutative monoids. Much of this foundational material is already available in the literature, but we have decided to offer a self-contained presentation more directly suited to our purposes. In log geometry, in an apparent contrast with toric geometry, the study of the *category* of monoids, and in particular of *homomorphisms of monoids*, plays a fundamental role. This difference was part of our motivation for including this material, and we hope our treatment may be of interest apart from its applications to log geometry per se. Thus Chapter I begins with the study of limits and colimits in the category of monoids, and in particular with the construction of pushouts, which are analogous to tensor products in the category of rings. We then discuss sets endowed with a monoid action (the analogs of modules in ring theory), ideals, localization, and the spectrum of a monoid, with its Zariski topology. After these preliminaries we turn to more familiar constructions in convex geometry, including basic results about finiteness, duality, and cones. Then we discuss monoid algebras and some facts about affine toric varieties. The final sections of Chapter I are devoted to a deeper study of properties of homomorphisms and actions of monoids, and in particular to certain analogs of flatness. Of particular importance is Kato's key concept of *exactness*, which we encounter in Section 1.1. An example of its importance is manifest in the "four point lemma" 4.2.16, where exactness is needed to make fiber products of logarithmically integral log schemes behave well. *Integrality* and *saturation* of morphisms, which we discuss next, are refinements of the notion of exactness. Theorem 4.7.2 reveals the structure of "critically" and "locally" exact homomorphisms and plays an important role throughout log geometry and this text. We finish by showing how locally exact homomorphisms can be made integral and saturated by a suitable base change, which can be viewed as a logarithmic version of semistable reduction. This material is more technical than the rest of our exposition and can be skipped over in a first reading.

Chapter II discusses sheaves of monoids on topological spaces. After disposing of some generalities, we define *monoschemes*, which are constructed by gluing together spectra of commutative monoids, just as schemes are constructed by gluing together spectra of commutative rings. Our monoschemes are sometimes called "schemes over $\mathbf{F}_1$" in the literature [12] and are generalizations of the *fans* used to construct toric varieties. We use this concept to construct *monoidal transformations* (blowups) for monoids (and monoschemes). The main application is Theorem 1.8.1, which explains how a homomorphism of monoids can be made locally exact by a monoidal transformation. Section 1.10 explains the *moment map* for a monoid scheme $\underline{\mathsf{A}}_{\mathsf{Q}}$, which gives a linearized model of the set of its $\mathbf{R}_{\geq}$-valued points. As an application, we show that the "positive part" of each fiber of a monoidal transformation is contractible. The remainder of Chapter II is devoted to Kato's important notions of *charts* and *coherence* for sheaves of monoids, which form the main technical link between logarithmic and toric geometry.

With the preliminaries well in hand, we are ready in Chapter III to turn to logarithmic geometry per se, including two variants of the standard theory:

idealized log schemes and *relatively coherent log structures*. We work with log structures in both the Zariski and étale topologies, since each has its own advantages and disadvantages, and explain the relation between the two. After giving the main definitions and basic constructions, we discuss some examples: log points and dashes, and the compactifying log structures coming from open immersions $U \to X$. We then describe in some detail a precursor of the notion of log structures, due (independently) to Deligne and Faltings. This notion, although less flexible and functorial than the point of view taken here, is convenient for describing the log structures that arise in the context of divisors with normal crossings and semistable reduction. It was in some sense already envisioned in the work of Friedman [20] and Steenbrink [73]. We then discuss *hollow* and especially *solid* log structures. In the first case, the log structure reflects the geometry of the part of a scheme that has been cut away, and in the second the log structure is tightly tied to the part of the geometry of the scheme which can be modelled by a toric structure. The notion of solidity of a log structure is closely related to, and helpful in, the study of Kato's notion of *log regularity*, which we discuss next. Finally, we briefly discuss *frames* for log structures, a weak version of charts that can be quite useful.

The remainder of Chapter III is devoted to the study of morphisms of log schemes, including the rather delicate construction of fibered products. *Exact* morphisms of log schemes play an especially important role, as well as the related notions of *integral* and *saturated* morphisms. We also study the logarithmic versions of immersions, inseparable morphisms, and *Kummer* and *small* morphisms, as well as *logarithmic blowups*.

Chapter IV is devoted to *logarithmic differentials* and *logarithmic smoothness*. We begin with a purely algebraic construction of Kähler differentials for (pre) log schemes, then explain its geometric meaning in terms of deformation theory. Next we discuss smoothness for logarithmic schemes, defined in terms of a logarithmic version of Grothendieck's infinitesimal lifting criterion. Although smooth morphisms in logarithmic geometry are much more complicated than in classical geometry, locally they admit nice toric models. As in the classical case, smoothness and regularity are related notions, the former being a "relative" version of the latter. We next discuss the more general notion of *logarithmic flatness*, which is quite useful but, as of this writing, technically challenging. We explore the relationships among the notions of flatness, smoothness, exactness, and integrality, extending in some cases the fundamental results of Kato.

In Chapter V we discuss topology and cohomology. To provide a geometric intuition, we begin with the construction of the *Betti realization* X_{log} of a log scheme X over $\mathbf{C}$. This is a topological space that comes with a natural proper map $\tau_X \colon X_{log} \to X_{an}$ which embodies the picture exemplified in Figure 1.1.1. We explain the definition and basic topological properties of Betti realizations, make them explicit for toric models, and show that the Betti realization of a smooth analytic space is a topological manifold with boundary. (We do not include the proof, but in fact the Betti realization of a smooth proper and exact morphism of log analytic spaces is a topological fibration of manifolds with boundary [57].) We then define the sheaf of rings $\mathcal{O}_X^{log}$ on X_{log}, which is obtained by adjoining logarithms of sections of $\mathcal{M}_X$ in a canonical way and which allows for a generalization of the familiar exponential sequence in classical complex analytic geometry. Our next main topic is logarithmic de Rham cohomology. We begin with an algebraic description of the *logarithmic de Rham complex* of a monoid algebra and some of the natural filtrations (defined by faces and ideals) it carries. Then we explain the sheafification and globalization of these constructions for log schemes. We give several versions of the logarithmic Poincaré lemma in the analytic setting, proving that analytic de Rham cohomology calculates the Betti cohomology of X_{log}. In the algebraic setting, we construct the Cartier isomorphism and the Cartier operator in positive characteristics, and explain how the Cartier operator relates to the restricted Lie-algebra structure on the logarithmic tangent sheaf. Finally we study algebraic de Rham cohomology in characteristic zero, concluding with some finiteness theorems and comparisons with analytic, and hence log Betti, cohomology.

Time and space constraints have prevented us from discussing many important topics which we had earlier hoped to include and for which we can only indicate some references in the literature. Some fundamental results not covered include the resolution of toric singularities [49],[58],[42], the cohomology of log blowups [39], and the fact that normal toric varieties are Cohen–Macaulay [36]. We have also had to omit examples of applications of log geometry and can only suggest that the reader look at work on the moduli of stable curves [47], on the logarithmic Riemann–Hilbert correspondence [40, 61], and on crystalline cohomology [37, 60], as a scattered set of examples.

1.4 Organization

The goals of this text are to introduce the reader to the basic ideas of log geometry and to provide a technical foundation for further work on the theory and its applications. These goals are somewhat contradictory, in that a good deal of the foundational material depends on the algebra of monoids and the geometry of convex bodies, the study of which can impede the momentum toward the ultimate goals coming from algebraic geometry. Although a fair amount of this material can be found in the literature, we have decided to treat it carefully here, partly because the author himself wanted to become comfortable with it and partly because the perspective from log geometry, in which homomorphisms play a central role, is not to be found in the standard texts. We have grouped nearly all this material in the first two chapters and consequently don't arrive at log geometry itself until Chapter III, potentially discouraging a reader eager to try out log geometry in some specific context. Such a reader may find it preferable to skip some of the earlier sections, returning to them as necessary. We hope our exposition will make this possible. In particular, the material on idealized monoids, idealized log schemes, and relative coherence, concepts whose ultimate utility has not yet been convincingly demonstrated, can be skipped on a first reading. Probably the same is true of monoschemes, which are really just an alternative to the classical theory of fans from toric geometry. Readers focused on the essence of log geometry could try reading only Sections 1.1, 4.1, and 4.2 of Chapter I, and then Sections 1.1 and 2.1 of Chapter II, before proceeding to Chapter III. Readers whose primary interest is convex rather than log geometry may find it interesting to concentrate on the material in Chapters I and II, since some of it may be new to them, especially Section 4 of Chapter I. Unfortunately, the key concept of logarithmic smoothness does not appear until well into Chapter IV; fortunately, this concept was already well explained in Kato's original paper [48]. In any case, we hope that impatient readers will find our treatment palatable even if they have not digested all the preceding material.

To facilitate flexibility in reading the text, we have tried to be careful with references. We use the same numbering scheme for definitions, theorems, remarks, etc. within each chapter. When referring to a result from a different chapter, we include the (roman numeral) chapter number in the reference; otherwise we omit it.

It is probably appropriate to remark on the writing style. We have attempted to include a considerable degree of detail, both in motivating and in defining concepts and in writing the proofs. Some readers, especially those familiar with the techniques of toric geometry, may consequently find the presentation ponderous. However, we found no alternative compatible with the goals of solidifying our understanding and of avoiding a plethora of errors, which would otherwise crop up not just in the proofs themselves, but also in statements of theorems and, worse, definitions. It seems easier for the reader to skip some arguments as s/he sees fit rather than to worry about errors hidden in unwritten proofs. Readers who feel the (understandable) desire for exercises can refrain from reading the proofs supplied and provide their own and/or content themselves with the search for errors, of which we fear many may remain. We would be grateful for notifications of any errors, which we hope eventually to correct on a web page available at `<https://math.berkeley.edu/~ogus/logpage.html>`.

1.5 Acknowledgements

Most of the material presented here is already in the literature in one form or another, often in several places. I have not made a systematic attempt to keep track of the proper original attributions. The main conceptual ideas of the form of logarithmic geometry treated here are due to L. Illusie, J.-M. Fontaine, and K. Kato; a precursor was developed independently by P. Deligne and by G. Faltings. In many places when I got stuck on basics I had invaluable help and guidance from G. Bergman, H. Lenstra, B. Sturmfels, C. Nakayama, and A. Abbes. I would also like to thank IRMAR and the IHES for their very generous support for this project. Many graduate students at U.C. Berkeley have attended courses on this material and helped to clarify the exposition; B. Conrad also gave a course at Stanford using a preliminary version of the manuscript and provided me with much meticulous and invaluable feedback. Many other mathematicians have also pointed out errors and misprints. O. Gabber's corrections and suggestions were particularly subtle and valuable. Special thanks go to Luc Illusie, who provided detailed advice, guidance, and important mathematical content throughout the planning and writing of this book; any defects in presentation or technical accuracy occur only when I failed to follow his advice. Most of the technical drawings were provided by J. Ogus, and the commutative diagrams were produced with Paul Taylor's Commutative Diagrams package. I am very appreciative of the patience, support, and encouragement provided by Kaitlin Leach, acquisition editor of Cambridge University Press, during the very long preparation of this manuscript.