I

The Geometry of Monoids

1 Basics on monoids

1.1 Limits in the category of monoids

A *monoid* is a triple $(M, \star, e_M)$ consisting of a set M, an associative binary operation $\star$, and a two-sided identity element e_M of M. A *homomorphism of monoids* is a function $\theta : M \to N$ such that $\theta(e_M) = e_N$ and $\theta(m \star m') = \theta(m) \star \theta(m')$ for any pair of elements m and m' of M. Note that, although the element e_M is the unique two-sided identity of M, compatibility of θ with e_M is not automatic from compatibility with $\star$. All monoids we consider here will be commutative unless explicitly noted otherwise, and we write **Mon** for the category of commutative monoids and homomorphisms of monoids.

We will often follow the common practice of writing M or $(M, \star)$ in place of $(M, \star, e_M)$ when there seems to be no danger of confusion. Similarly, if a and b are elements of a monoid $(M, \star, e_M)$, we will often write ab (or $a + b$) for $a \star b$, and 1 (or 0) for e_M.

The most basic example of a monoid is the set **N** of natural numbers, with addition as the monoid law. If M is any monoid and $m \in M$, there is a unique monoid homomorphism $\mathbf{N} \to M$ sending 1 to m; thus **N** is the free monoid with generator 1. More generally, if S is any set, the set $\mathbf{N}^{(S)}$ of functions $I : S \to \mathbf{N}$ such that $I_s = 0$ for almost all s, endowed with the pointwise addition of functions as a binary operation, is the free (commutative) monoid with basis $S \subseteq N^{(S)}$. Thus the functor $S \mapsto \mathbf{N}^{(S)}$ is left adjoint to the forgetful functor from the category of monoids to the category of sets.

Arbitrary (projective) limits exist in the category of monoids, and their formation commutes with the forgetful functor to the category of sets. In particular, the intersection of a set of submonoids of M is again a submonoid; hence if S is a subset of M, the intersection of all the submonoids of M that contain

S is the smallest submonoid of M containing S, the *submonoid of M generated by S*. If there exists a finite subset S of M that generates M, one says that M is *finitely generated* as a monoid.

Arbitrary colimits (inductive limits) of monoids also exist. Direct sums are easy to construct: the direct sum $\bigoplus M_i$ of a family $\{M_i : i \in I\}$ of monoids is the submonoid of the product $\prod_i M_i$ consisting of those elements $m.$ such that $m_i = 0$ for almost all i. The general construction is more difficult, and we will first investigate quotients and equivalence relations in the category of monoids.

Let $\theta\colon P \to Q$ be a homomorphism of monoids. Note that the kernel $\theta^{-1}(0)$ of θ is not very useful: for example, the kernel of the homomorphism $\theta\colon \mathbf{N} \oplus \mathbf{N} \to \mathbf{N}$ sending (a, b) to $a + b$ is just $\{(0, 0)\}$, but θ is not injective. Instead we consider the set $E(\theta)$ of pairs $(p_1, p_2) \in P \times P$ such that $\theta(p_1) = \theta(p_2)$, an equivalence relation on P. The fact that θ is also a homomorphism of monoids implies that $E(\theta)$ is a submonoid of $P \times P$. An equivalence relation on P that is also a submonoid of $P \times P$ is called a *congruence* (or *congruence relation*) on P. One checks easily that if E is a congruence relation on P, then the set P/E of equivalence classes has a unique monoid structure making the projection $P \to P/E$ a monoid homomorphism. Thus there is a dictionary between congruence relations on P and isomorphism classes of surjective monoid homomorphisms $P \to P'$. The following proposition, whose proof is immediate, summarizes these considerations.

Proposition 1.1.1. *Let P be a monoid.*

1. *Let $\pi\colon P \to Q$ be a surjective homomorphism of monoids, and let E be the equalizer of the two maps $P \times P \underset{p_2}{\overset{p_1}{\rightrightarrows}} P \xrightarrow{\pi} Q$, i.e., $E := P \times_Q P$.*
   (a) *E is a congruence relation on P.*
   (b) *Q is the coequalizer of the two maps $E \to P \times P \underset{p_2}{\overset{p_1}{\rightrightarrows}} P$. Thus, the diagram*

$$\begin{array}{ccc} E & \longrightarrow & P \\ \downarrow & & \downarrow \\ P & \longrightarrow & Q \end{array}$$

   is cocartesian, as well as cartesian.
2. *Let $E \subseteq P \times P$ be a congruence relation on P, let $Q := P/E$ be the set of equivalence classes, and let $\pi\colon P \to Q$ be the function taking an element of P to the equivalence class containing it.*

(a) *There is a unique monoid structure on Q such that $\pi\colon P \to Q$ is a monoid homomorphism.*
(b) *The inclusion $e : E \to P\times P$ is the equalizer of the two homomorphisms*

$$P \times P \underset{p_2}{\overset{p_1}{\rightrightarrows}} P \xrightarrow{\pi} Q,$$

and Q is the coequalizer of the two homomorphisms

$$E \xrightarrow{e} P \times P \underset{p_2}{\overset{p_1}{\rightrightarrows}} P.$$

Thus, the diagram of (1b) is cartesian and cocartesian.

The passage from E to Q induces a bijection from the set of congruence relations on P to the set of isomorphism classes of surjective homomorphisms whose domain is P. □

In the terminology of [2, 10.3 and 10.8, Exp. I], Proposition 1.1.1 says that every surjective homomorphism of monoids is an "effective epimorphism," and every congruence relation E is an "effective equivalence relation."

Remark 1.1.2. If $P \to Q$ is surjective and $Q' \to Q$ is any homomorphism, then the pullback map $P \times_Q Q' \to Q'$ is again surjective. This implies that $P \to Q$ and E are in fact "universally effective." On the other hand, not every epimorphism in the category of monoids is surjective. In fact, a homomorphism of monoids is universally an epimorphism if and only if it is surjective.

The intersection of a family of congruence relations is a congruence relation, and hence it makes sense to speak of the *congruence relation generated by a subset of* $P \times P$. One says that a congruence relation E is *finitely generated* if there is a finite subset S of $P \times P$ that generates E as a congruence relation; this does not imply that S generates E as a monoid.

Here is a useful description of the congruence relation generated by a subset of $P \times P$.

Proposition 1.1.3. *Let P be a (commutative) monoid.*

1. *An equivalence relation $E \subseteq P \times P$ is a congruence relation if and only if $(a + p, b + p) \in E$ whenever $(a, b) \in E$ and $p \in P$.*
2. *If S is a subset of $P \times P$, let*

$$S_P := \{(a + p, b + p) : (a, b) \in S, p \in P\}.$$

Then the congruence relation E generated by S is the equivalence relation generated by S_P. Explicitly, E is the union of the diagonal with the set of pairs (x, y) for which there exists a finite sequence $(p_0, \ldots, p_n)$ with $p_0 = x$

and $p_n = y$ such that, for every $i > 0$, either (p_{i-1}, p_i) or (p_i, p_{i-1}) belongs to S_P.

Proof Let E be an equivalence relation on P that is stable under addition by elements of the diagonal. Suppose that (a, b) and $(c, d) \in E$. Then $(a+c, b+c) \in E$ and $(c + b, d + b) \in E$, and since P is commutative and E is transitive, $(a + c, b + d) \in E$. Thus E is closed under addition. Since E contains the diagonal, the identity element $(0, 0)$ of $P \times P$ belongs to E, so E is a submonoid of $P \times P$, hence a congruence relation. Conversely, if E is a congruence relation, then $(p, p) \in E$ for every $p \in E$; hence $(a + p, b + p) \in E$ whenever $(a, b) \in E$, This proves (1). For (2), let E denote the congruence relation generated by S and let E' denote the equivalence relation generated by S_P. Since $S_P \subseteq E$ and E is an equivalence relation, it follows that $E' \subseteq E$. The associative law implies that S_P is stable under addition by elements of the diagonal of $P \times P$. Hence if $(p_0, \ldots, p_n)$ is a sequence such that (p_{i-1}, p_i) or $(p_i, p_{i-1}) \in S_P$ for all $i > 0$, then $(p_0 + p, \ldots, p_n + p)$ shares the same property. Thus if $(x, y) \in E'$ and $p \in P$, then $(x + p, y + p) \in E'$. Then it follows from (1) that E' is a congruence relation, and so $E' = E$. □

Remark 1.1.4. If P is an abelian group and $E \subseteq P \times P$ is a congruence relation on P, then the image of E under the homomorphism $h \colon P \oplus P \to P$ sending (p_1, p_2) to $p_2 - p_1$ is a subgroup K of P, and $E = h^{-1}(K)$. Conversely the inverse image under h of any subgroup of P is a congruence relation on P. This defines a bijective correspondence between the subgroups of P and the congruence relations on P.

If θ_1 and θ_2 are monoid homomorphisms $P \to Q$, one can construct the *coequalizer* of θ_1 and θ_2 as the quotient of Q by the congruence relation on Q generated by the set of pairs $(\theta_1(p), \theta_2(p))$ for $p \in P$.

The existence of arbitrary colimits follows from the existence of direct sums and coequalizers of pairs of morphisms by the following standard construction (see also [2, 2.3, Exp. I]). Let $\{P_i, \theta_a\}$ be a functor from a small category I to the category of monoids, where i ranges over the objects of I and a over the arrows $a \colon i(a) \to j(a)$ of I. Since the category of monoids has direct sums, we can form $Q := \oplus\{P_i : i \in Ob(I)\}$ and $R := \oplus\{P_{i(a)}, a \in Arr(I)\}$, with canonical homomorphisms

$$\{u_i \colon P_i \to Q : i \in Ob(I)\} \quad \text{and} \quad \{u_a \colon P_{i(a)} \to R : a \in Arr(I)\}.$$

Then there are unique homomorphisms $\theta_1, \theta_2 \colon R \to Q$ such that, for all a,

$\theta_1 \circ u_a = u_{i(a)}$ and $\theta_2 \circ u_a = u_{j(a)} \circ \theta_a$:

$$
\begin{array}{ccc}
P_{i(a)} & \xrightarrow{u_a} & R \\
 & \searrow^{u_{i(a)}} & \downarrow \theta_1 \\
 & & Q
\end{array}
\qquad
\begin{array}{ccc}
R & \xleftarrow{u_a} & P_{i(a)} \\
\downarrow \theta_2 & & \downarrow \theta_a \\
Q & \xleftarrow{u_{j(a)}} & P_{j(a)}.
\end{array}
$$

The colimit of the functor $\{P_i, \theta_a\}$ is the coequalizer of θ_1 and θ_2.

A *presentation* of a monoid M is a coequalizer diagram

$$L_1 \rightrightarrows L_0 \longrightarrow M$$

with L_0 and L_1 free. It is equivalent to the data of a map from a set I to M whose image generates M and a map from a set J to $\mathbf{N}^{(I)} \times \mathbf{N}^{(I)}$ whose image generates the congruence relation on $\mathbf{N}^{(I)}$ defined by the surjective monoid map $\mathbf{N}^{(I)} \to M$ corresponding to the set map $I \to M$. The monoid M is said to be *of finite presentation* if it admits a presentation with L_0 and L_1 free and finitely generated. We shall see in Theorem 2.1.7 that in fact every finitely generated (commutative) monoid is of finite presentation.

The *amalgamated sum* $Q_1 \xrightarrow{v_1} Q \xleftarrow{v_2} Q_2$ of a pair of monoid morphisms $u_i \colon P \to Q_i$, often denoted simply by $Q_1 \oplus_P Q_2$, is the colimit of the diagram $Q_1 \xleftarrow{u_1} P \xrightarrow{u_2} Q_2$. That is, the pair (v_1, v_2) universally makes the diagram

$$
\begin{array}{ccc}
P & \xrightarrow{u_1} & Q_1 \\
u_2 \downarrow & & \downarrow v_1 \\
Q_2 & \xrightarrow{v_2} & Q
\end{array}
\qquad (1.1.1)
$$

commute. This amalgamated sum can be viewed as the pushout of u_1 along u_2 or the pushout of u_2 along u_1. It can also be viewed as the coequalizer of the two maps $(u_1, 0)$ and $(0, u_2)$ from P to $Q_1 \oplus Q_2$.

The following proposition describes the pushout $Q_1 \oplus_P Q_2$ explicitly. Its calculation is considerably simplified if one of the monoids in question is a group. (See Proposition 4.6.1 for a generalization.)

Proposition 1.1.5. *Let $u_i \colon P \to Q_i$ be a pair of monoid morphisms, let Q be their amalgamated sum, as in Diagram 1.1.1, and let E be the congruence relation on $Q_1 \oplus Q_2$ given by the natural surjection $Q_1 \oplus Q_2 \to Q$.*

1. *Let S be the set of pairs*

$$((q_1, q_2), (q_1', q_2')) \in (Q_1 \oplus Q_2) \times (Q_1 \oplus Q_2)$$

such that there exists a $p \in P$ such that $q_1' = u_1(p) + q_1$ and $q_2 = u_2(p) + q_2'$. Then E is the set of pairs

$$(a, b) \in (Q_1 \oplus Q_2) \times (Q_1 \oplus Q_2)$$

such that there exists a sequence $(r_0, \ldots, r_n)$ in $Q_1 \oplus Q_2$ such that $(a, b) = (r_0, r_n)$ and such that (r_i, r_{i+1}) belongs to S if i is even and to $S^t := \{(a, b) : (b, a) \in S\}$ if i is odd.

2. *Let E' be the set of pairs $((q_1, q_2), (q_1', q_2'))$ of elements of $Q_1 \oplus Q_2$ such that there exist p and p' in P with $q_1 + u_1(p') = q_1' + u_1(p)$ and $q_2 + u_2(p) = q_2' + u_2(p')$. Then E' is a congruence relation on $Q_1 \oplus Q_2$ containing E, and if any of P, Q_1, or Q_2 is a group, then $E = E'$.*
3. *If P is a group, then two elements of $Q_1 \oplus Q_2$ are congruent modulo E if and only if they lie in the same orbit of the action of P on $Q_1 \oplus Q_2$ defined by $p(q_1, q_2) = (q_1 + u_1(p), q_2 + u_2(-p))$.*
4. *If P and Q_i are groups, then so is $Q_1 \oplus_P Q_2$, which is in fact just the amalgamated sum in the category of abelian groups.*

Proof To prove (1), observe first that S is stable under the action of the diagonal of $(Q_1 \oplus Q_2) \times (Q_1 \oplus Q_2)$ and contains this diagonal. Then, by Proposition 1.1.3, the congruence relation R generated by S is the set of pairs (a, b) such that there exists a sequence $(r_0, \ldots, r_n)$ with $r_0 = a$ and $r_n = b$ and such that each pair (r_i, r_{i+1}) belongs to S or to S^t. Note however that if (r_{i-1}, r_i) and (r_i, r_{i+1}) both belong to S or to S^t, then so does (r_{i-1}, r_{i+1}), so the sequence can be shortened. Note further that if (r_0, r_1) belongs to S^t then (r_0, r_0, r_1) satisfies the description in (1). This shows that the set described in (1) really is a congruence relation. Since E contains S, and is in fact the smallest such congruence relation, it follows that $E = R$.

To prove (2), note first that the set E' is evidently symmetric and reflexive. To prove its transitivity, let us say that a pair (a, b) in $P \times P$ "links" a pair of elements (q_1, q_2) and (q_1', q_2') of $Q_1 \oplus Q_2$ if $q_1 + u_1(b) = q_1' + u_1(a)$ and $q_2 + u_2(a) = q_2' + u_2(b)$. One checks immediately that if (a, b) links (q_1, q_2) and (q_1', q_2') and (a', b') links (q_1', q_2') and (q_1'', q_2''), then $(a + a', b + b')$ links (q_1, q_2) and (q_1'', q_2''). Moreover, if (a, b) links (q_1, q_2) and (q_1', q_2') then, for any $(\tilde{q}_1, \tilde{q}_2) \in Q_1 \oplus Q_2$, (a, b) links $(q_1 + \tilde{q}_1, q_2 + \tilde{q}_2)$ and $(q_1' + \tilde{q}_1, q_2' + \tilde{q}_2)$. Then by Proposition 1.1.3, E' is a congruence relation on $Q_1 \oplus Q_2$. Furthermore, if $p \in P$, $(p, 0)$ links $(u_1(p), 0)$ and $(0, u_2(p))$, and since E is the congruence relation generated by such pairs, $E \subseteq E'$. If P or either Q_i is a group, then

$v := v_i \circ u_i$ factors through the group Q^* of invertible elements of Q. If (a, b) links (q_1, q_2) and (q'_1, q'_2), we find that

$$\begin{aligned} v_1(q_1) + v_2(q_2) + v(a + b) &= v_1(q_1 + u_1(b)) + v_2(q_2 + u_2(a)) \\ &= v_1(q'_1 + u_1(a)) + v_2(q'_2 + u_2(b)) \\ &= v_1(q'_1) + v_2(q'_2) + v(a + b). \end{aligned}$$

Since $v(a + b) \in Q^*$, it follows that

$$v_1(q_1) + v_2(q_2) = v_1(q'_1) + v_2(q'_2).$$

Thus $E' \subseteq E$. This proves (2), and (3) and (4) are immediate consequences. □

Example 1.1.6. Taking $Q_2 = 0$ in Proposition 1.1.5 one obtains the *cokernel* of the morphism $u_1 : P \to Q_1$, or, equivalently, the coequalizer of u_1 and the zero mapping $P \to Q_1$. If P is a submonoid of Q, one writes $Q \to Q/P$ for the cokernel of the inclusion $P \to Q$, and it follows from (2) of the proposition that two elements q and q' of Q have the same image in Q/P if and only if there exist p and p' in P such that $q + p = q' + p'$. For example, the cokernel of the diagonal embedding $\mathbf{N} \to \mathbf{N} \oplus \mathbf{N}$ is the homomorphism

$$\mathbf{N} \oplus \mathbf{N} \to \mathbf{Z} : (a, b) \mapsto a - b.$$

Note that Q/P can be zero even if P is a proper submonoid of Q; this holds, for example, if P is the submonoid of $Q := \mathbf{N} \oplus \mathbf{N}$ generated by $(1, 0)$ and $(1, 1)$. If P' is a submonoid of Q containing P, then P'/P is a submonoid of Q/P and the natural map $(Q/P)/(P'/P) \to Q/P'$ is an isomorphism.

1.2 Monoid actions

If S is a set, then the set of functions from S to itself forms a (not necessarily commutative) monoid End(S) under composition. If Q is a monoid, an *action of Q on S* is a monoid homomorphism θ_S from Q to End(S). In this context we often write the monoid law of Q multiplicatively, and, if $q \in Q$ and $s \in S$, we write qs for $\theta_S(q)(s)$. A *Q-set* is a set endowed with an action of Q, and $\mathbf{Ens}_Q$ will denote the category of Q-sets, with the evident notion of morphism. If S is a Q-set and $s \in S$, the image of the map $Q \to S$ sending q to qs is the minimal Q-stable subset of S containing s, called the *trajectory* of s in S.

A *basis* for a Q-set S is a map of sets $i: T \to S$ such that the induced map $Q \times T \to S: (q, t) \mapsto qi(t)$ is bijective. If such a basis exists we say that S is a *free Q-set*. A free Q-set with basis $T \to S$ satisfies the usual universal property of a free object: every map from T to the set underlying a Q-set S'

extends uniquely to a morphism of Q-sets $S \to S'$. If T is any set and if $Q \times T$ is endowed with the action ρ defined by $\rho(q')(q,t) = (q'q,t)$, then the map $T \to Q \times T$ sending t to $(1,t)$ is a basis. Thus the functor taking a set T to the free Q-set $Q \times T$ is left adjoint to the forgetful functor from the category of Q-sets to the category of sets. Note that if G is a group and S is a G-set, then S has a basis as a G-set if and only if the action is free in the sense that $gs = s$ implies $g = 1$. This equivalence is not true for general monoids.

The category $\mathbf{Ens}_Q$ of Q-sets admits arbitrary projective limits, and their formation commutes with the forgetful functor to the category of sets. This is a formal consequence of the fact that the forgetful functor $\mathbf{Ens}_Q \to \mathbf{Ens}$ has a left adjoint. In particular, if S and T are Q-sets, then Q acts on $S \times T$ by $q(s,t) := (qs,qt)$, and this action makes $S \times T$ the product of S and T in $\mathbf{Ens}_Q$.

Colimits in $\mathbf{Ens}_Q$ also exist. The direct sum of a family $S_i : i \in I$ is just the disjoint union with the evident Q-action. To understand the construction of quotients in the category $\mathbf{Ens}_Q$, note that if $\pi\colon S \to T$ is a surjective map of Q-sets, then the corresponding equivalence relation $E \subseteq S \times S$ is a Q-subset of $S \times S$; such an equivalence relation is called a *congruence relation* on S. Conversely, if E is any congruence relation on S, then there is a unique Q-set structure on S/E such that the projection $S \to S/E$ is a morphism of Q-sets. When $S = Q$ acting regularly on itself, the notion of a congruence relation on Q as a monoid coincides with the notion of a congruence relation as a Q-set, thanks to Proposition 1.1.3. Furthermore, the analog of statement (2) of Proposition 1.1.3 holds for Q-sets, and in particular the equivalence relation generated by a subset of $S \times S$ that is stable under the diagonal action of Q is already a congruence relation. If u and v are two morphisms $S' \to S$, the coequalizer of u and v is the quotient of S by the congruence relation generated by $\{(u(s'), v(s')) : s' \in S'\}$. The existence of general colimits follows.

If S and T are Q-sets, the set $\mathrm{Hom}_Q(S,T)$ has a natural action of Q, given by $(qh)(s) := qh(s) = h(qs)$ for $h\colon S \to T$, $q \in Q$, and $s \in S$ There is also a tensor product construction for Q-sets. If S, T, and W are Q-sets, then a *Q-bimorphism* $S \times T \to W$ is by definition a function $\beta\colon S \times T \to W$ such that $\beta(qs,t) = \beta(s,qt) = q\beta(s,t)$ for any $(s,t) \in S \times T$ and $q \in Q$. The *tensor product of S and T* is the universal Q-bimorphism $S \times T \to S \otimes_Q T$. To construct it, begin by regarding $S \times T$ as a Q-set via its action on S, so that $q(s,t) := (qs,t)$, and consider the equivalence relation R on $S \times T$ generated by the set of pairs

$$((qs,t),(s,qt)) \in (S \times T) \times (S \times T) \text{ for } q \in Q, s \in S, t \in T.$$

Note that this set of pairs is stable under the action of Q, since if $q' \in Q$, and $s' := q's$ then $((q'qs,t),(q's,qt)) = ((qs',t),(s',qt))$. It follows that the equivalence relation R is a congruence relation. Then the projection $\pi\colon S \times T \to$

$(S \times T)/R$ is a Q-bimorphism and satisfies the universal mapping property of the tensor product. If Q is a (commutative) group, then $S \otimes_Q T$ can be constructed in the usual way as the orbit space of the action of Q on $S \times T$ given by $q(s,t) := (qs, q^{-1}t)$.

In general, one has a natural isomorphism of Q-sets

$$\mathrm{Hom}_Q(S \otimes_Q T, W) \cong \mathrm{Hom}_Q(S, \mathrm{Hom}_Q(T, W)),$$

taking a Q bimorphism β to the Q-morphism γ given by $\gamma(s)(t) := \beta(s,t)$. It follows formally that, for fixed T, the functor $S \mapsto S \otimes_Q T$ commutes with colimits.

Let $\theta \colon P \to Q$ be a homomorphism of monoids. Then θ defines an action of P on Q given by $pq := \theta(p)q$. If S is a P-set, the tensor product $Q \otimes_P S$ has the natural action of Q, with $q(q' \otimes s) = (qq' \otimes s)$, and the map $S \to Q \otimes_P S$ sending s to $1 \otimes s$ is a morphism of P-sets over the homomorphism θ. If $\theta_i \colon P \to Q_i$ for $i = 1, 2$ is a pair of monoid homomorphisms, then there is a unique monoid structure on $Q_1 \otimes_P Q_2$ such that

$$(q_1 \otimes q_2)(q_1' \otimes q_2') = (q_1 q_1' \otimes q_2 q_2'),$$

and this is also the unique monoid structure for which the natural maps $Q_i \to Q_1 \otimes_P Q_2$ are homomorphisms. It can be checked that this monoid structure makes $Q_1 \otimes_P Q_2$ into the amalgamated sum of Q_1 and Q_2 along P.

Remark 1.2.1. We have seen that, for a fixed Q-set T, the functor $S \mapsto S \otimes_Q T$ commutes with colimits. It is perhaps no surprise that it does not commute with limits in general. We want to emphasize that this functor need not even commute with finite products, even if T is free. Indeed, if T has basis Λ, then $S \otimes_Q T \cong S \times \Lambda$, and if the cardinality of Λ is greater than one, the functor $S \mapsto S \times \Lambda$ does not commute with products. This fact complicates the calculation of tensor products from generators and relations. Indeed, suppose that $F \to S$ is a surjective morphism and $E := F \times_S F$ is the corresponding equivalence relation on F, where F is free. Then $F \to S$ is the coequalizer of the two maps $E \rightrightarrows F$, and, since $\otimes_Q T$ commutes with colimits, it follows that $F \otimes_Q T \to S \otimes_Q T$ is the coequalizer of $E \otimes_Q T \rightrightarrows F \otimes_Q T$. However, the natural map

$$(F \times F) \otimes_Q T \to (F \otimes_Q T) \times (F \otimes_Q T)$$

is not an isomorphism, and the image of $E \otimes_Q T$ in $(F \otimes_Q T) \times (F \otimes_Q T)$ might not be an equivalence relation. Thus one is left with the often challenging problem of computing the congruence relation it generates.

Definition 1.2.2. *Let Q be a monoid and let S be a Q-set. The transporter of S is the category $\mathcal{T}_Q S$ whose objects are the elements of S and for which the*

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morphisms from an object s to an object t are the elements q of Q such that $qs = t$, with composition given by the monoid law of Q. The transporter of a monoid Q is the transporter of Q regarded as a Q-set, and is denoted simply by $\mathcal{T}Q$.

Associated with the category $\mathcal{T}_Q S$ is a partially ordered set that is worth making explicit.

Definition 1.2.3. *Let Q be a monoid and S a Q-set. If s and t are elements of S, we write $s \leq t$ if there exists a $q \in Q$ such that $qs = t$, and $s \sim t$ if $s \leq t$ and $t \leq s$.*

It is clear that $s \leq w$ if $s \leq t$ and $t \leq w$ and that $s \leq s$ for every $s \in S$. Thus the relation $\leq$ defines a preorder on S. The relation $\sim$ is a congruence relation on S, and the relation $\leq$ on the quotient $S/\!\sim$ is a partial order. We shall use this notion especially when $S = Q$ with the regular representation. Since $\sim$ is a congruence relation, it follows from Proposition 1.1.3 that $Q/\!\sim$ inherits a monoid structure.

1.3 Integral, fine, and saturated monoids

If M is any commutative monoid, there is a universal homomorphism λ_M from M to a group M^{gp}. That is, M^{gp} is a group, $\lambda_M \colon M \to M^{\mathrm{gp}}$ is a homomorphism of monoids, and any homomorphism from M to a group factors uniquely through λ_M. Thus, the functor $M \mapsto M^{\mathrm{gp}}$ is the left adjoint of the inclusion functor from the category of groups to the category of monoids; since it has a right adjoint, it automatically commutes with the formation of direct limits. In fact, M^{gp} can be identified with the cokernel (Example 1.1.6) of $M \oplus M = M \times M$ by the diagonal, and λ_M with the composite of $(\mathrm{id}_M, 0)$ and the projection $M \times M \to (M \times M)/\Delta_M$. One can also construct M^{gp} as the set of equivalence classes of pairs (x, y) of elements of M, where (x, y) is equivalent to (x', y') if and only if there exists $z \in M$ such that $x + y' + z = x' + y + z$. The explicit description of the equivalence relation in Example 1.1.6 shows that the two constructions are in fact the same. One writes $x - y$ for the equivalence class containing (x, y), and then $(x - y) + (x' - y') = (x + x') - (y + y')$.

If M is a monoid, let M^* denote the set of all $m \in M$ such that there exists an $n \in M$ such that $m + n = 0$. Then M^* forms a submonoid of M. It is in fact a subgroup—the largest subgroup of M. Any homomorphism from a group to M factors uniquely through M^*, so that $M \mapsto M^*$ is right adjoint to the inclusion functor from groups to monoids. We call M^* the *group of units* of M; it acts naturally on M by translation. If G is any subgroup of M, the orbit space M/G