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HARDY SPACES

The theory of Hardy spaces is a cornerstone of modern analysis. It combines techniques from functional analysis, the theory of analytic functions, and Lesbesgue integration to create a powerful tool for many applications, pure and applied, from signal processing and Fourier analysis to maximum modulus principles and the Riemann zeta function.

This book, aimed at beginning graduate students, introduces and develops the classical results on Hardy spaces and applies them to fundamental concrete problems in analysis. The results are illustrated with numerous solved exercises which also introduce subsidiary topics and recent developments. The reader's understanding of the current state of the field, as well as its history, are further aided by engaging accounts of the key players and by the surveys of recent advances (with commented reference lists) that end each chapter. Such broad coverage makes this book the ideal source on Hardy spaces.

Nikolaï Nikolski is Professor Emeritus at the Université de Bordeaux working primarily in analysis and operator theory. He has been co-editor of four international journals and published numerous articles and research monographs. He has also supervised some 30 PhD students, including three Salem Prize winners. Professor Nikolski was elected Fellow of the AMS in 2013 and received the Prix Ampère of the French Academy of Sciences in 2010.

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Less is more

Robert Browning,
"Andrea del Sarto," 1855

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Preface

The *introduction to Hardy spaces* proposed in this book covers the basic techniques of modern analysis, conceived and developed at the beginning of the twentieth century over a very short period (a kind of "Silver Age" for mathematical analysis; Exercise 1: which was the "Golden Age"?), by a talented group of mathematical geniuses including Henri Lebesgue, Frigyes Riesz, G. H. Hardy, Andrey Kolmogorov, and Norbert Wiener. Over time, this cluster of ideas became the source of extremely powerful techniques for a variety of applications: from Fourier series to the Wiener theory of stationary filtering, not to mention the Euler ζ function and the Riemann hypothesis.

The contents of this text correspond to a course at the "Master 2" level given several times during the years 1990–2010 at the University of Bordeaux 1, and represent an introduction and invitation to the entire domain of modern analysis. The book is devoted to a multi-faceted subject: it involves harmonic analysis (since it concerns a unitary representation of the group $\mathbb{Z}$), but also complex analysis (as we restrict ourselves most often to the semigroup $\mathbb{Z}_+$), the theory of operators (by the nature of the representation, but also by a hidden universality that we will explore in future volumes), as well as the theme of signals and filtering, with a bit of number theory thrown in. It is for this superposition of major disciplines of mathematics (more a "roundabout" than a "crossroads") that the subject can be described as "classical" ("classical" ≠ "old-fashioned"!). The conjunction between the different facets of the subject is most fruitful and successful in the *Hilbert* framework of the spaces $L^2(\mathbb{T}, \mu)$; this is why we have developed the theory, and its applications, principally in the space H^2 (which is also closely linked with H^1 and H^∞), whereas the other H^p spaces appear only occasionally.

The prerequisites are a standard course in integration and functional analysis (or in Hilbert/Banach spaces) along with a few notions of complex analysis. A summary/reminder of all the necessary information (as well as certain notations) are gathered in the appendices at the end of the book. Within the text, we include a large number of historical details – on the subjects developed, their founders, and the diverse circumstances of their creation. We hope that this will help the reader to better understand Hardy spaces, along with the dramaturgy of mathematics (and mathematical life).[1]

Each chapter contains exercises and their solutions (75 in total) at different levels: to use a musical analogy from Glazman and Lyubich (1969), from exercises on open strings up to virtuoso pieces using double harmonics ("double flageolet tones").

Each chapter concludes with a section entitled "Notes and Remarks" which discusses the history of the main subjects of the chapter, certain recent results, and (at times) the open questions; this discussion is sometimes addressed to non-novice readers.

The reader will rapidly become aware that this text contains only a few elementary aspects of the techniques of harmonic analysis, linked particularly with an approach to Hardy spaces via complex analysis. Even if at times we delve into quite refined questions of analysis (such as the geometry of finite bases, in Chapter 4), our text is not meant to be a research monograph, but more a source of basic knowledge. This is why "less is more." Nonetheless, in principle, students reaching the end of the book should be capable of tackling independent research projects (the author can affirm this from experience). For such an endeavor they will need the aid of experts, but this can be found in the dozens of existing monographs devoted to Hardy spaces and the "hard analysis" that was developed around them. Several are mentioned at the end of Chapter 1, in the section Notes and Remarks 1.9. Good luck!

[1] The biographical details – which, given the technical and financial constraints, are sketched here at best – are drawn from various sources, notably the *MacTutor* website of the University of St Andrews (Scotland), www-history.mcs.st-and.ac.uk/, and the free encyclopedia *Wikipedia*, https://en.wikipedia.org/wiki/.

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I thank Éditions Belin (and especially the editor responsible for this work, C. Counillon) for accepting my project and seeing it through to fruition.

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