

A Student's Guide to General Relativity

This compact guide presents the key features of General Relativity, to support and supplement the presentation in mainstream, more comprehensive undergraduate textbooks, or as a recap of essentials for graduate students pursuing more advanced studies. It helps students plot a careful path to understanding the core ideas and basic techniques of differential geometry, as applied to General Relativity, without overwhelming them. While the guide doesn't shy away from necessary technicalities, it emphasizes the essential simplicity of the main physical arguments. Presuming a familiarity with Special Relativity (with a brief account in an appendix), it describes how general covariance and the equivalence principle motivate Einstein's theory of gravitation. It then introduces differential geometry and the covariant derivative as the mathematical technology which allows us to understand Einstein's equations of General Relativity. The book is supported by numerous worked examples and exercises, and important applications of General Relativity are described in an appendix.

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Before thir eyes in sudden view appear
 The secrets of the hoarie deep, a dark
 Illimitable Ocean without bound,
 Without dimension, where length, breadth, & highth,
 And time and place are lost;
 [...]

Into this wilde Abyss,
 The Womb of nature and perhaps her Grave,
 Of neither Sea, nor Shore, nor Air, nor Fire,
 But all these in thir pregnant causes mixt
 Confus'dly, and which thus must ever fight,
 Unless th' Almighty Maker them ordain
 His dark materials to create more Worlds,
 Into this wild Abyss the warie fiend
 Stood on the brink of Hell and look'd a while,
 Pondering his Voyage: for no narrow frith
 He had to cross.

John Milton, *Paradise Lost*, II, 890–920

But in the dynamic space of the living Rocket,
 the double integral has a different meaning. To integrate
 here is to operate on a rate of change so that time falls away:
 change is stilled . . . 'Meters per second' will integrate
 to 'meters.' The moving vehicle is frozen, in space,
 to become architecture, and timeless.
 It was never launched.
 It will never fall.

Thomas Pynchon, *Gravity's Rainbow*

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Preface

This introduction to General Relativity (GR) is deliberately short, and is tightly focused on the goal of introducing differential geometry, then getting to Einstein's equations as briskly as possible.

There are four chapters:

Chapter 1 – Introduction and Motivation.

Chapter 2 – Vectors, Tensors, and Functions.

Chapter 3 – Manifolds, Vectors, and Differentiation.

Chapter 4 – Physics: Energy, Momentum, and Einstein's Equations.

The principal mathematical challenges are in Chapters 2 and 3, the first of which introduces new notations for possibly familiar ideas. In contrast, Chapters 1 and 4 represent the connection to physics, first as motivation, then as payoff. The main text of the book does not cover Special Relativity (SR), nor does it cover applications of GR to any significant extent. It is useful to mention SR, however, if only to fix notation, and it would be perverse to produce a book on GR without a mention of at least *some* interesting metrics, so both of these are discussed briefly in appendices.

When it comes down to it, there is not a huge volume of material that a physicist must learn before they gain a technically adequate grasp of Einstein's equations, and a long book can obscure this fact. We must learn how to describe coordinate systems for a rather general class of spaces, and then learn how to differentiate functions defined on those spaces. With that done, we are over the threshold of GR: we can define interesting functions such as the Energy-Momentum tensor, and use Einstein's equations to examine as many applications as we need, or have time for.

This book derives from a ten-lecture honours/masters course I have delivered for a number of years in the University of Glasgow. It was the first of a pair

of courses: this one was ‘the maths half’, which provided most of the maths required for its partner, which focused on various applications of Einstein’s equations to the study of gravity. The course was a compulsory one for most of its audience: with a smaller, self-selecting class, it might be possible to cover the material in less time, by compressing the middle chapters, or assigning readings; with a larger class and a more leisurely pace, we could happily spend a lot more time at the beginning and end, discussing the motivation and applications.

In adapting this course into a book, I have resisted the temptation to expand the text at each end. There are already many excellent but heavy tomes on GR – I discuss a few of them in Section 1.4.2 – and I think I would add little to the sum of world happiness by adding another. There are also shorter treatments, but they are typically highly mathematical ones, which don’t amuse everyone. Relativity, more than most topics, benefits from your reading multiple introductions, and I hope that this book, in combination with one or other of the mentioned texts, will form one of the building blocks in your eventual understanding of the subject.

As readers of any book like this will know, a lecture course has a *point*, which is either the exam at the end, or another course that depends on it. This book doesn’t have an exam, but in adapting it I have chosen to act as if it did: the book (minus appendices) has the same material as the course, in both selection and exclusion, and has the same practical goal, which is to lead the reader as straightforwardly as is feasible to a working understanding of the core mathematical machinery of GR. Graduate work in relativity will of course require mining of those heavier tomes, but I hope it will be easier to explore the territory after a first brisk march through it. The book is not designed to be dipped into, or selected from; it should be read straight through. Enjoy the journey.

Another feature of lecture courses and of Cambridge University Press’s *Student’s Guides*, which I have carried over to this book, is that they are bounded: they do not have to be complete, but can freely refer students to other texts, for details of supporting or corroborating interest. I have taken full advantage of this freedom here, and draw in particular on Schutz’s *A First Course in General Relativity* (2009), and to a somewhat lesser extent on Carroll’s *Spacetime and Geometry* (2004), aligning myself with Schutz’s approach except where I have a positive reason to explain things differently. This book is not a ‘companion’ to Schutz, and does not assume you have a copy, but it is deliberately highly compatible with it. I am greatly indebted both to these and to the other texts of Section 1.4.2.

In writing the text, I have consistently aimed for succinctness; I have generally aimed for one precise explanation rather than two discursive ones, while remembering that I am writing a physics text, and not a maths one. And in line with the intention to keep the destination firmly in mind, there are rather few major excursions from our route. The book is intended to be usable as a primary resource for students who need or wish to know some GR but who will not (yet) specialise in it, and as a secondary resource for students starting on more advanced material.

The text includes a number of exercises, and the density of these reflects the topics where my students had most difficulty. Indeed, many of the exercises, and much of the balance of the text, are directly derived from students' questions or puzzles. Solutions to these exercises can be downloaded at www.cambridge.org/gray.



Throughout the book, there are various passages, and a couple of complete sections, marked with 'dangerous bend' signs, like this one. They indicate supplementary details, material beyond the scope of the book which I think may be nonetheless interesting, or extra discussion of concepts or techniques that students have found confusing or misunderstandable in the past. If, again, this book had an exam, these passages would be firmly out of bounds.

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