1

Introduction

What is the problem that General Relativity (GR) is trying to solve? Section 1.1 introduces the principle of general covariance, the relativity principle, and the equivalence principle, which between them provide the physical underpinnings of Einstein's theory of gravitation.

We can examine some of these points a second time, at the risk of a little repetition, in Section 1.2, through a sequence of three thought experiments, which additionally bring out some immediate consequences of the ideas. It's rather a matter of taste, whether you regard the thought experiments as motivation for the principles, or as illustrations of them.

The remaining sections in this chapter are other prefatory remarks, about 'natural units' (in which the speed of light c and the gravitational constant G are both set to 1), and pointers to a selection of the many textbooks you may wish to consult for further details.

1.1 Three Principles

Newton's second law is

$$\mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}, \tag{1.1}$$

which has the special case, when the force $\mathbf{F}$ is zero, of $\mathrm{d}\mathbf{p}/\mathrm{d}t = 0$: The momentum is a conserved quantity in any force-free motion. We can take this as a statement of Newton's first law. In the standard example of first-year physics, of a puck moving across an ice rink or an idealised car moving along an idealised road, we can start to calculate with this by attaching a rectilinear coordinate system S to the rink or to the road, and discovering that

$$\mathbf{F} = m\mathbf{a} = m\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}, \tag{1.2}$$

from which we can deduce the constant-acceleration equations and, from that, all the fun and games of Applied Maths 1.

Alternatively, we could describe a coordinate system S' rotating about the origin of our rectilinear one with angular speed Ω, in which

$$\mathbf{F}' = m\mathbf{a}' = -m\Omega \times (\Omega \times \mathbf{r}') - 2m\Omega \times \frac{\mathrm{d}\mathbf{r}'}{\mathrm{d}t}, \tag{1.3}$$

and then derive the equations of constant acceleration from that. Doing so would not be wrong, but it would be perverse, because the underlying physical statement is the same in both cases, but the expression of it is more complicated in one frame than in the other. Put another way, Eq. (1.1) is physics, but the distinction between Eqs. (1.2) and (1.3) is merely mathematics.

This is a more profound statement than it may at first appear, and it can be dignified as

> *The principle of general covariance:* All physical laws must be invariant under all coordinate transformations.

A putative physical law that depends on the details of a particular frame – which is to say, a particular coordinate system – is one that depends on a mathematical detail that has no physical significance; we must rule it out of consideration as a physical law. Instead, Eq. (1.1) is a relation between two geometrical objects, namely a momentum vector and a force vector, and this illustrates the geometrical approach that we follow in this text: a physical law must depend only on geometrical objects, independent of the frame in which we realise them. In order to do calculations with it, we need to pick a particular frame, but that is incidental to the physical insight that the equation represents. The geometrical objects that we use to model physical quantities are vectors, one-forms, and tensors, which we learn about in Chapter 2.

It is necessary that the differentiation operation in Eq. (1.1) is also frame-independent. Right now, this may seem too obvious to be worth drawing attention to, but in fact a large part of the rest of this text is about defining differentiation in a way that satisfies this constraint. You may already have come across this puzzle, if you have studied the convective derivative in fluid mechanics or the tensor derivative in continuum mechanics, and you will have had hints of it in learning about the various forms of the Laplacian in different coordinate systems. See Section 1.3 for a preview.

It is also fairly obvious that Eq. (1.2) is a simpler expression than Eq. (1.3). This observation is not of merely aesthetic significance, but it prompts us to discover that there is a large class of frames where the expression of Newton's second law takes the same simple form as Eq. (1.2); these frames are the frames that are moving with respect to S with a constant velocity $\mathbf{v}$, and we call each of the members of this class an *inertial frame*. In each inertial frame, motion is simple and, moreover, each inertial frame is related to another in a simple way: namely the *galilean transformation* in the case of pre-relativistic physics, and the *Lorentz transformation* in the case of Special Relativity (SR).

The fact that the observational effects of Newton's laws are the same in each inertial frame means that we cannot tell, from observation only of dynamical phenomena within the frame, which frame we are in. Put less abstractly, you can't tell whether you're moving or stationary, without looking outside the window and detecting movement relative to some other frame. Inertial frames thus have, or at least can be taken to have, a special status. This special status turns out, as a matter of observational fact, to be true not only of dynamical phenomena dependent on Newton's laws, but of all physical laws, and this also can be elevated to a principle.

> *The principle of relativity (RP):* (a) All true equations in physics (i.e., all 'laws of nature', and not only Newton's first law) assume the same mathematical form relative to all local inertial frames. Equivalently, (b) no experiment performed wholly within one local inertial frame can detect its motion relative to any other local inertial frame.

If we add to this principle the axiom that the speed of light is infinite, we deduce the galilean transformation; if we instead add the axiom that the speed of light is a frame-independent constant (an axiom that turns out to be amply confirmed by observation), we deduce the Lorentz transformation and Special Relativity. In SR, remember, we are obliged to talk of a four-dimensional coordinate frame, with one time and three space dimensions.

General Relativity – Einstein's theory of gravitation – adds further significance to the idea of the inertial frame. Here, an inertial frame is a frame in which SR applies, and thus the frame in which the laws of nature take their corresponding simple form. This definition, crucially, applies even in the presence of large masses where (in newtonian terms) we would expect to find a gravitational force. The frames thus picked out are those which are in *free fall*, either because they are in deep space far from any masses, or because they are (attached to something that is) moving under the influence of 'gravitation' alone. I put 'gravitation' in scare quotes because it is part of the point of GR to demote gravitation from its newtonian status as a distinct physical force to a status as a mathematical fiction – a conceptual convenience – which is no more real than centrifugal force.

The first step of that demotion is to observe that the force of gravitation (I'll omit the scare quotes from now on) is strangely independent of the

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nature of the things that it acts upon. Imagine a frame sitting on the surface of the Earth, and in it a person, a bowl of petunias, and a radio, at some height above the ground: we discover that, when they are released, each of them will accelerate at the same rate towards the floor (Galileo is supposed to have demonstrated this same thing using the Tower of Pisa, careless of the health and safety of passers-by). Newton explains this by saying that the force of gravitation on each object is proportional to its gravitational mass (the gravitational ‘charge’, if you like); and the acceleration of each object, in response to that force, is proportional to its inertia, which is proportional to its inertial mass. Newton doesn’t put it in those terms, of course, but he also fails to explain why the gravitational and inertial masses, which *a priori* have nothing to do with each other, turn out experimentally to be *exactly* proportional to each other, even though the person, the plant, the plantpot, and the radio broadcasting electromagnetic waves all exhibit very different physical properties.

Now imagine this same frame – or, for the sake of concreteness and the containment of a breathable atmosphere, a spacecraft – floating in space. Since spacecraft, observer, petunias, and radio are all equally floating in space, none will move with respect to another (or, if they are initially moving, they will continue to move with constant relative velocity). That is, Newton’s laws work in their simple form in this frame, which we can therefore identify as an inertial frame.

If, now, we turn on the spacecraft’s engines, then the spacecraft will accelerate, but the objects within it will not, until the spacecraft collides with them, and starts to accelerate them by pushing them with what we will at that point decide to call the cabin floor. Crucially – and, from this point of view, obviously – the sequence of events here is independent of the details of the structure of the ceramic plantpot, the biology of the observer and the petunias, and the electronic intricacies of the radio. If the spacecraft continues to accelerate at, say, $9.81\,\mathrm{m\,s^{-2}}$, then the objects now firmly on the cabin floor will experience a continuous force of one standard Earth gravity, and observers within the cabin will find it difficult to tell whether they are in an accelerating spacecraft or in a uniform gravitational field.

In fact we can make the stronger statement – and this is another physical statement which has been verified to considerable precision in, for example, the Eötvös experiments – that the observers will find it *impossible* to tell the difference between acceleration and uniform gravitation; and this is a third remark that we can elevate to a physical principle.

> *The Equivalence Principle (EP):* Uniform gravitational fields are equivalent to frames that accelerate uniformly relative to inertial frames.

The EP is closely related to the observation that gravitational and inertial mass are strictly proportional; Rindler, for example, refers to this as the 'weak' equivalence principle (see Section 4.2.2).

We can summarise where we have got to as follows: (i) the principle of general covariance thus constrains the possible forms of statements of physical law, (ii) the EP and RP point to a privileged status of inertial frames in our search for further such laws, (iii) the RP gives us a link to the physics that we already know at this stage, and (iv) the EP gives us a link to the 'gravitational fields' that we want to learn more about.

These three principles make a variety of physical and mathematical points.

- The principle of general covariance restricts the category of mathematical statements that we are prepared to countenance as possible descriptions of nature. It says something about the relationship between physics and mathematics.
- The RP is either, in version (b) above, a straightforwardly physical statement or, in version (a), a physical statement in mathematical form. It picks out inertial frames as having a special status, and by saying that all inertial frames have equal status, it restricts the transformation between any pair of frames.
- The EP is also a physical statement. As we will examine further in Chapter 4, it further constrains the set of 'special' inertial frames, while retaining the idea that these inertial frames are physically indistinguishable, and exploring the constraints that that equivalence imposes.

By a 'physical statement' I mean a statement that picks out one of multiple mathematically consistent possibilities, and says that *this one* is the one that matches our universe. Mathematically, we *could* have a universe in which the galilean transformation works for all speeds, and the speed of light is infinite; but we don't.

Most of the statements in this section can be quibbled with, sometimes with great sophistication. The statement of the RP is quoted with minor adaptation from Barton (1999), who discusses the principle at book length in the context of SR. The wording of the EP is from Schutz (2009, §5.1), but Rindler (2006) discusses this with characteristic precision in his early chapters (distinguishing weak, strong, and semistrong variants of the EP), and Misner, Thorne and Wheeler (1973, §§7.2–7.3) discuss it with characteristic vividness. There is a minor industry devoted to the precise physical content of the EP and the principle of general covariance, and to their logical relationship to Einstein's theory of gravity. This industry is discussed at substantial length by Norton (1993), and subsequent texts quoting it, but it does not seem to contribute usefully to an elementary discussion such as this one, and I have thought it best to keep the account in this section as compact and as straightforward as possible, while noting that there is much more one can go on to think about.

1.2 Some Thought Experiments on Gravitation

At the risk of some repetition, we can make the same points again, and make some further interesting deductions, through a sequence of thought experiments.

1.2.1 The Falling Lift

Recall from SR that we may define an inertial frame to be one in which Newton's laws hold, so that particles that are not acted on by an external force move in straight lines at a constant velocity. In Misner, Thorne, and Wheeler's words, inertial frames and their time coordinates are defined so that motion looks simple. This is also the case if we are in a box far away from any gravitational forces, we may identify that as a *local inertial frame* (we will see the significance of the word 'local' later in the chapter). Another way of removing gravitational forces – less extreme than going into deep space – is to put ourselves in free fall. Einstein asserted that these two situations are indeed fully equivalent, and defined an inertial frame as one in free fall.

Objects at rest in an inertial frame – in either of the equivalent situations of being far away from gravitating matter or freely falling in a gravitational field – will stay at rest. If we accelerate the box-cum-inertial-frame, perhaps by attaching rockets to its 'floor', then the box will accelerate but its contents won't; they will therefore move towards the floor at an increasing speed, from the point of view of someone in the box.[1] This will happen irrespective of the mass or composition of the objects in the box; they will all appear to increase their speed at the same rate.

Note that I am carefully *not* using the word 'accelerate' for the change in speed of the objects in the box with respect to that frame. We reserve that word for the physical phenomenon measured by an accelerometer, and the result of a real force, and try to avoid using it (not, I fear, always successfully) to refer

[1] By 'point of view' I mean 'as measured with respect to a reference frame fixed to the box', but such circumlocution can distract from the point that this is an *observation* we're talking about – we can see this happening.

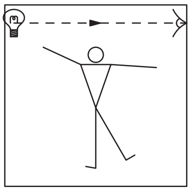

Figure 1.1 A floating box.

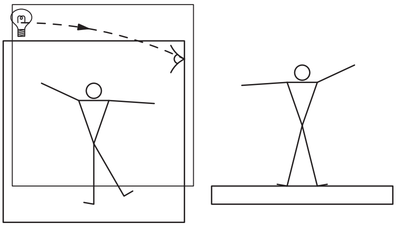

Figure 1.2 A free-fall box.

to the second derivative of a position. Depending on the coordinate system, the one does not always imply the other, as we shall see later.

This is very similar to Galileo's observation that all objects fall under gravity at the same rate, irrespective of their mass or composition. Einstein supposed that this was not a coincidence, and that there was a deep equivalence between acceleration and gravity (we shall see later, in Chapter 4, that the force of gravity that we feel while standing in one place is the result of us being accelerated away from the path we would have if we were in free fall). He raised this to the status of a postulate: the Equivalence Principle.

Imagine being in a box floating freely in space, and imagine shining a torch horizontally across it from one wall to the other (Figure 1.1). Where will the beam end up? Obviously, it will end up at a point on the wall directly opposite the torch. There's nothing exotic about this. The EP tells us that the same must happen for a box in free fall. That is, a person inside a falling lift would observe the torch beam to end up level with the point at which it was emitted, in the (inertial) frame of the lift. This is a straightforward and unsurprising use of the EP. How would this appear to someone watching the lift fall?

Since the light takes a finite time to cross the lift cabin, the spot on the wall where it strikes will have dropped some finite (though small) distance, and so will be lower than the point of emission, in the frame of someone watching this from a position of safety (Figure 1.2). That is, this non-free-fall observer will measure the light's path as being curved in the gravitational field. Even massless light is affected by gravity. [Exercise 1.1]

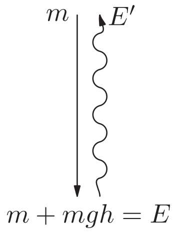

Figure 1.3 The Pound-Rebka experiment.

1.2.2 Gravitational Redshift

Imagine dropping a particle of mass m through a distance h. The particle starts off with energy m ($E = mc^2$, with $c = 1$; see Section 1.4.1), and ends up with energy $E = m + mgh$ (see Figure 1.3). Now imagine converting all of this energy into a single photon of energy E, and sending it up towards the original position. It reaches there with energy E', which we convert *back* into a particle.[2] Now, either we have invented a perpetual motion machine, or else $E' = m$:

$$E' = m = \frac{E}{1 + gh}, \tag{1.4}$$

and we discover that a photon loses energy as a necessary consequence of climbing through a gravitational field, and as a consequence of our demand that energy be conserved.

This energy loss is termed *gravitational redshift*, and it (or rather, something very like it) has been confirmed experimentally, in the 'Pound-Rebka experiment'. It's also sometimes referred to as 'gravitational doppler shift', but inaccurately, since it is not a consequence of relative motion, and so has nothing to do with the doppler shift that you are familiar with.

Light, it seems, can tell us about the gravitational field it moves through.

1.2.3 Schild's Photons

Imagine firing a photon, of frequency f, from an event A to an event B spatially located directly above it in a gravitational field (see Figure 1.4). As we discovered in the previous section, the photon will be redshifted to a new frequency f'. After some number of periods n, we repeat this, and send up another photon (between the points marked A' and B' on the space-time diagram).

[2] As described, this is kinematically impossible, since we cannot do this and conserve momentum, but we can imagine sending distinct particles back and forth, conserving just energy; this would have an equivalent effect, but be more intricate to describe precisely.

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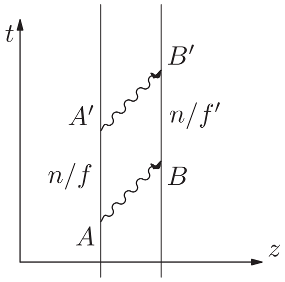

Figure 1.4 Schild's photons.

Photons are a kind of clock, in that the interval between 'wavecrests', $1/f$, forms a kind of 'tick'. The length of this tick will be measured to have different numerical values in different frames, but the start and end of the interval nonetheless constitute two frame-independent events.

Presuming that the source and receiver are not in relative motion, the intervals AB and $A'B'$ will be the same (I've drawn these as straight lines on the diagram, but the argument doesn't depend on that). However, the intervals AA' and BB' comprise the same number n of periods, which means that the intervals in time, n/f and n/f', as measured by local clocks, are *different*. That is, we have not constructed the parallelogram we might have expected, and have therefore discovered that the geometry of this space-time is not flat geometry we might have expected, and that this is purely as a result of the presence of the gravitational field through which we are sending the photons.

Finding out more about this geometry is what we aim to do in this text.

The 'Schild's photons' argument, and a version of the gravitational redshift argument, first appeared in Schild (1962), where both are presented in careful and precise detail. The subtleties are important, but the arguments in the sections earlier in this chapter, though slightly schematic, contain the essential intuition. Schild's paper also includes a thoughtful discussion of what parts of GR are and are not addressed by experiment.

1.2.4 Tides and Geodesic Deviation (and Local Frames)

Consider two particles, A and B, both falling towards the earth, with their height from the centre of the earth given by $z(t)$ (Figure 1.5). They start off level with each other and separated by a horizontal distance $\xi(t)$.

From the diagram, the separation $\xi(t)$ is proportional to $z(t)$, so that $\xi(t) = kz(t)$, for some constant k. The gravitational force on a particle of mass m at altitude z is $F = GMm/z^2$, thus

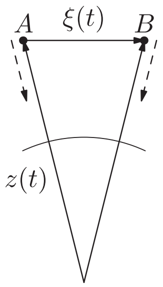

Figure 1.5 Two falling particles.

$$\frac{d^2\xi}{dt^2} = k\frac{d^2z}{dt^2} = -k\frac{F}{m} = -k\frac{GM}{z^2} = -\xi\frac{GM}{z^3}.$$

This tells us that the inertial frames attached to these freely falling particles approach each other at an increasing speed (that is, they ‘accelerate’ towards each other in the sense that the second derivative of their separation is non-zero, but since they are in free fall, there is no physical acceleration that an observer in the frame would feel as a push).

If A and B are two observers in inertial frames (or inertial spacecraft), then we have said that they cannot distinguish between being in space far from any gravitating masses, and being in free fall near a large mass. If instead they found themselves at opposite ends of a giant free-falling spacecraft, then they would find themselves drifting closer to each other as the spacecraft fell, in apparent violation of Newton’s laws. Is there a contradiction here?

No. The EP as quoted in Section 1.1 talked of *uniform* gravitational fields, which this is not. Also, both the RP of that section, and the discussion in Section 1.2.1, talked of *local* inertial frames. A lot of SR depends on inertial frames having infinite extent: if I am an inertial observer, then any other inertial observer must be moving at a constant velocity with respect to me. In GR, in contrast, an inertial frame is a local approximation (indeed it is fully accurate only at a point, an important issue we will return to later), and if your measurement or experiment is sufficiently extended in space or time, or if your instruments are sufficiently accurate, then you will be able to detect tidal forces in the way that A and B have done in this thought experiment.

If A and B are plummeting down lift shafts, in free fall, on opposite sides of the earth, then they are inertial observers, but they are ‘accelerating’ with respect to one another. This means that, if I am one of these inertial observers, then (presuming I do not have more pressing things to worry about) I cannot use SR to calculate what the other inertial observer would measure in their frame, nor calculate what I would measure if I observed a bit of physics that I understand, which is happening in the other inertial observer’s frame.