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PART VII: EXTENSIONS OF AD, MODELS WITH CHOICE

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A BRIEF HISTORY OF DETERMINACY

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§1. **Introduction.** Determinacy axioms are statements to the effect that certain games are *determined*, in that each player in the game has an optimal strategy. The commonly accepted axioms for mathematics, the Zermelo-Fraenkel axioms with the Axiom of Choice (ZFC; *cf.* [Jec03, Kun11]), imply the determinacy of many games that people actually play. This applies in particular to many **games of perfect information**, games in which the players alternate moves which are known to both players, and the outcome of the game depends only on this list of moves, and not on chance or other external factors. Games of perfect information which must end in finitely many moves are determined. This follows from the work of Ernst Zermelo [Zer13], Dénes Kőnig [Kőn27] and László Kálmar [Kal28], and also from the independent work of John von Neumann and Oskar Morgenstern (in their 1944 book, reprinted as [vNM04]).

As pointed out by Stanisław Ulam [Ula60], determinacy for games of perfect information of a fixed finite length is essentially a theorem of logic. If we let $x_1,y_1,x_2,y_2,\dots,x_n,y_n$ be variables standing for the moves made by players player I (who plays $x_1,\dots,x_n$) and player II (who plays $y_1,\dots,y_n$), and A (consisting of sequences of length $2n$) is the set of runs of the game for which player I wins, the statement

$$\exists x_1 \forall y_1 \dots \exists x_n \forall y_n \langle x_1, y_1, \dots, x_n, y_n\rangle \in A$$

essentially asserts that the first player has a winning strategy in the game, and its negation,

$$\forall x_1 \exists y_1 \dots \forall x_n \exists y_n \langle x_1, y_1, \dots, x_n, y_n\rangle \notin A$$

essentially asserts that the second player has a winning strategy.[1]

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[1] If there exists a way of choosing a member from each nonempty set of moves of the game, then these statements are actually equivalent to the assertions that the corresponding strategies exist. Otherwise, in the absence of the Axiom of Choice the statements above can hold without the corresponding strategy existing.

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We let ω denote the set of natural numbers $0, 1, 2, \ldots$; for brevity we will often refer to the members of this set as "integers". Given sets X and Y, ${}^{X}Y$ denotes the set of functions from X to Y. The **Baire space** is the space ${}^{\omega}\omega$, with the product topology. The Baire space is homeomorphic to the space of irrational real numbers (*cf.*, *e.g.*, [Mos09, p. 9]), and we will often refer to its members as "reals" (though in various contexts the Cantor space ${}^{\omega}2$, the set of subsets of ω ($\wp(\omega)$) and the set of infinite subsets of ω ($[\omega]^{\omega}$) are all referred to as "the reals").

Given $A \subseteq {}^{\omega}\omega$, we let $\mathrm{G}_{\omega}(A)$ denote the game of perfect information of length ω in which the two players collaborate to define an element f of ${}^{\omega}\omega$ (with player I choosing $f(0)$, player II choosing $f(1)$, player I choosing $f(2)$, and so on), with player I winning a run of the game if and only if f is an element of A. A game of this type is called an **integer game**, and the set A is called the **payoff set**. A **strategy** in such a game for player I (player II) is a function Σ with domain the set of sequences of integers of even (odd) length such that for each $a \in \mathrm{dom}(\Sigma)$, $\Sigma(a)$ is in ω. A run of the game (partial or complete) is said to be **according to** a strategy Σ for player I (player II) if every initial segment of the run of odd (nonzero even) length is of the form $a^\frown\langle\Sigma(a)\rangle$ for some sequence a. A strategy Σ for player I (player II) is a **winning strategy** if every complete run of the game according to Σ is in (out of) A. We say that a set $A \subseteq {}^{\omega}\omega$ is **determined** (or the corresponding game $\mathrm{G}_{\omega}(A)$ is determined) if there exists a winning strategy for one of the players. These notions generalize naturally for games in which players play objects other than integers (*e.g.*, **real games**, in which they play elements of ${}^{\omega}\omega$) or games which run for more than ω many rounds (in which case player I typically plays at limit stages).

The study of determinacy axioms concerns games whose determinacy is neither proved nor refuted by the Zermelo-Fraenkel axioms ZF (without the Axiom of Choice). Typically such games are infinite. Axioms stating that infinite games of various types are determined were studied by Stanisław Mazur, Stefan Banach and Ulam in the late 1920s and early 1930s; were reintroduced by David Gale and Frank Stewart [GS53] in the 1950s and again by Jan Mycielski and Hugo Steinhaus [MS62] in the early 1960s; gained interest with the work of David Blackwell [Bla67] and Robert Solovay in the late 1960s; and attained increasing importance in the 1970s and 1980s, finally coming to a central position in contemporary set theory.

Mycielski and Steinhaus introduced the Axiom of Determinacy (AD), which asserts the determinacy of $\mathrm{G}_{\omega}(A)$ for all $A \subseteq {}^{\omega}\omega$. Work of Banach in the 1930s shows that AD implies that all sets of reals satisfy the property of Baire. In the 1960s, Mycielski and Stanisław Świerczkowski proved that AD implies that all sets of reals are Lebesgue measurable, and Mycielski showed that AD implies countable choice for reals. Together, these results show that determinacy provides a natural context for certain areas of mathematics, notably analysis, free of the paradoxes induced by the Axiom of Choice.

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Unaware of the work of Banach, Gale and Stewart [GS53] had shown that AD contradicts ZFC. However, their proof used a wellordering of the reals given by the Axiom of Choice, and therefore did not give a nondetermined game of this type with definable payoff set. Starting with Banach's work, many simply definable payoff sets were shown to induce determined games, culminating in D. Anthony Martin's celebrated 1974 result [Mar75] that all games with Borel payoff set are determined. This result came after Martin had used measurable cardinals to prove the determinacy of games whose payoff set is an analytic sets of reals.

The study of determinacy gained interest from two theorems in 1967, the first due to Solovay and the second to Blackwell. Solovay proved that under AD, ω_1 (the least uncountable ordinal) is a measurable cardinal, setting off a study of strong Ramsey properties on the ordinals implied by determinacy axioms. Blackwell used open determinacy (proved by Gale and Stewart) to reprove a classical theorem of Kazimierz Kuratowski. This also led to the application, by John Addison, Martin, Yiannis Moschovakis and others, of stronger determinacy axioms to produce structural properties for definable sets of reals. These axioms included the determinacy of $\undertilde{\mathbf{\Delta}}^1_n$ sets of reals, for $n \geq 2$, statements which would not be proved consistent relative to large cardinals until the 1980s.

The large cardinal hierarchy was developed over the same period, and came to be seen as a method for calibrating consistency strength. In the 1970s, various special cases of $\undertilde{\mathbf{\Delta}}^1_2$ determinacy were located on this scale, in terms of the large cardinals needed to prove them. Determining the consistency (relative to large cardinals) of forms of determinacy at the level of $\undertilde{\mathbf{\Delta}}^1_2$ and beyond would take the introduction of new large cardinal concepts. Martin (in 1978) and W. Hugh Woodin (in 1984) would prove $\undertilde{\mathbf{\Pi}}^1_2$-determinacy and $\mathsf{AD}^{\mathbf{L}(\mathbb{R})}$ respectively, using hypotheses near the very top of the large cardinal hierarchy. In a dramatic development, the hypotheses for these results would be significantly reduced through work of Woodin, Martin and John Steel. The initial impetus for this development was a seminal result of Matthew Foreman, Menachem Magidor and Saharon Shelah which showed, assuming the existence of a supercompact cardinal, that there exists a generic elementary embedding with well-founded range and critical point ω_1. Combined with work of Woodin, this yielded the Lebesgue measurability of all sets in the inner model $\mathbf{L}(\mathbb{R})$ from this hypothesis. Shelah and Woodin would reduce the hypothesis for this result further, to the assumption that there exist infinitely many Woodin cardinals below a measurable cardinal.

Woodin cardinals would turn out to be the central large cardinal concept for the study of determinacy. Through the study of tree representations for sets of reals, Martin and Steel would show that $\undertilde{\mathbf{\Pi}}^1_{n+1}$-determinacy follows from the existence of n Woodin cardinals below a measurable cardinal, and that this hypothesis is not sufficient to prove stronger determinacy results for the

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projective hierarchy. Woodin would then show that the existence of infinitely many Woodin cardinals below a measurable cardinal implies $\mathsf{AD}^{\mathbf{L}(\mathbb{R})}$, and he would locate the exact consistency strengths of $\utilde{\mathbf{\Delta}}^1_2$-determinacy and $\mathsf{AD}^{\mathbf{L}(\mathbb{R})}$ at one Woodin cardinal and ω Woodin cardinals respectively.

In the aftermath of these results, many new directions were developed, and we give only the briefest indication here. Using techniques from inner model theory, the exact consistency strengths of many determinacy hypotheses were established. Using similar techniques, it has been shown that almost every natural statement (*i.e.*, not invented specifically to be a counterexample) implies directly those determinacy hypotheses of lesser consistency strength. For instance, by Kurt Gödel's Second Incompleteness Theorem, ZFC cannot prove that the AD holds in $\mathbf{L}(\mathbb{R})$, as the latter implies the consistency of the former. Empirically, however, every natural extension T of ZFC of sufficient consistency strength (*i.e.*, such that Peano Arithmetic does not prove the consistency of of T from the consistency of ZF+ AD) does appear to imply that AD holds in $\mathbf{L}(\mathbb{R})$. This sort of phenomenon is taken by some as evidence that the statement that AD holds in $\mathbf{L}(\mathbb{R})$, and other determinacy axioms, should be counted among the true statements extending ZFC (*cf.*, *e.g.*, [KW]).

The history presented here relies heavily on those given by Jackson [Jac10], Kanamori [Kan95, Kan03], Moschovakis [Mos09], Neeman [Nee04] and Steel [Ste08B]. As the title suggests, this is a selective and abbreviated account of the history of determinacy. We have omitted many interesting topics, including, *e.g.*, Blackwell games [Bla69, Mar98, MNV03] and proving determinacy in second-order arithmetic [LSR87, LSR88, KW10].

§2. Early developments. The first published paper in mathematical game theory appears to be Zermelo's paper [Zer13] on chess. Although he noted that his arguments apply to all games of reason not involving chance, Zermelo worked under two additional chess-specific assumptions. The first was that the game in question has only finitely many states, and the second was that an infinite run of the game was to be considered a draw. Zermelo specified a condition which is equivalent to the existence of a strategy in such a game guaranteeing a win within a fixed number of moves, as well as another condition equivalent to the existence of a strategy guaranteeing that one will not lose within a given fixed number of moves. His analysis implicitly introduced the notions of **game tree**, **subtree** of a game tree, and **quasi-strategy**.[2]

The paper states indirectly, but does not quite prove, or even define, the statement that in any game of perfect information with finitely many possible positions such that infinite runs of the game are draws, either one player

[2]As defined above, a strategy for a given player specifies a move in each relevant position; a quasi-strategy merely specifies a set of acceptable moves. The distinction is important when the Axiom of Choice fails, but is less important in the context of Zermelo's paper.

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has a strategy that guarantees a win, or both players have strategies that guarantee at least a draw. A special case of this fact is determinacy for games of perfect information of a fixed finite length, which is sometimes called Zermelo's Theorem.

König [Kőn27] applied the fundamental fact now known as **Kőnig's Lemma** to the study of games, among other topics. While Kőnig's formulation was somewhat different, his lemma is equivalent to the assertion that every infinite finitely branching tree with a single root has an infinite path (a path can be found by iteratively choosing any successor node such that the tree above that node is infinite). Extending Zermelo's analysis to games in which infinitely many positions are possible while retaining the condition that each player has only finitely many options at each point, Kőnig used the statement above to prove that in such a game, if one player has a strategy (from a given point in the game) guaranteeing a win, then he can guarantee victory within a fixed number of moves. The application of Kőnig's Lemma to the study of games was suggested by von Neumann.

Kálmar [Kal28] took the analysis a step further by proving Zermelo's Theorem for games with infinitely many possible moves in each round. His arguments proceeded by assigning transfinite ordinals to nodes in the game tree, a method which remains an important tool in modern set theory. Kálmar explicitly introduced the notion of a winning strategy for a game, although his strategies were also quasi-strategies as above. In his analysis, Kálmar introduced a number of other important technical notions, including the notion of a **subgame** (essentially a subtree of the original game tree), and classifying strategies into those which depend only on the current position in the game and those which use the history of the game so far.[3]

Games of perfect information for which the set of infinite runs is divided into winning sets for each player appear in a question by Mazur in the Scottish Book, answered by Banach in an entry dated 4 August 1935 (*cf.* [Mau81, p. 113]). Following up on Mazur's question (still in the Book), Ulam asked about games where two players collaborate to build an infinite sequence of 0's and 1's by alternately deciding each member of the sequence, with the winner determined by whether the infinite sequence constructed falls inside some predetermined set E. Essentially raising the issue of determinacy for arbitrary $\mathrm{G}_\omega(E)$, Ulam asked: for which sets E does the first player (alternately, the second player) have a winning strategy? (§ 2.1 below has more on the Banach-Mazur game.)

Games of perfect information were formally defined in 1944 by von Neumann and Morgenstern [vNM04]. Their book also contains a proof that games of perfect information of a fixed finite length are determined (p. 123).

Infinite games of perfect information were reintroduced by Gale and Stewart [GS53], who were unaware of the work of Mazur, Banach and Ulam (Gale,

[3] *Cf.* [SW01] for much more on these papers of Zermelo, Kőnig and Kálmar.

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personal communication). They showed that a nondetermined game can be constructed using the Axiom of Choice (more specifically, from a wellordering of the set of real numbers).[4] They also noted that the proof from the Axiom of Choice does not give a definable undetermined game, and raised the issue of whether determinacy might hold for all games with a suitably definable payoff set. Toward this end, they introduced a topological classification of infinite games of perfect information, defining a game (or the set of runs of the game which are winning for the first player) to be **open** if all winning runs for the first player are won at some finite stage (*i.e.*, if, whenever $\langle x_0, x_1, x_2, \ldots\rangle$ is a winning run of the game for the first player, there is some n such that the first player wins all runs of the game extending $\langle x_0, \ldots, x_n\rangle$). Using this framework, they proved a number of fundamental facts, including the determinacy of all games whose payoff set is a Boolean combination of open sets (*i.e.*, in the class generated from the open sets by the operations of finite union, finite intersection and complementation). The determinacy of open games would become the basis for proofs of many of the strongest determinacy hypotheses. Gale and Stewart also asked a number of important questions, including the question of whether all Borel games are determined (to be answered positively by Martin [Mar75] in 1974).[5] Classifying games by the definability of their payoff sets would be an essential tool in the study of determinacy.

2.1. Regularity properties. Early motivation for the study of determinacy was given by its implications for regularity properties for sets of reals. In particular, determinacy of certain games of perfect information was shown to imply that every set of reals has the property of Baire and the perfect set property, and is Lebesgue measurable.[6] These three facts themselves each contradict the Axiom of Choice. We will refer to Lebesgue measurability, the property of Baire and the perfect set property as the **regularity properties**, the fact that there are other regularity properties notwithstanding.

[4]Given a set Y, we let AC_Y denote the statement that whenever $\{X_a : a \in Y\}$ is a collection of nonempty sets, there is a function f with domain Y such that $f(a) \in X_a$ for all $a \in Y$. Zermelo's **Axiom of Choice** (AC) [Zer04] is equivalent to the statement that AC_Y holds for all sets Y. A linear ordering $\leq$ of a set X is a **wellordering** if every nonempty subset of X has a $\leq$-least element. The Axiom of Choice is equivalent to the statement that there exist wellorderings of every set.

Kőnig's Lemma is a weak form of the Axiom of Choice and cannot be proved in ZF (*cf.* [Lév79, Exercise IX.2.18]).

[5]The **Borel** sets are the members of the smallest class containing the open sets and closed under the operations of complementation and countable union. The collection of Borel sets is generated in ω_1 many stages from these two operations. A natural process assigns a measure to each Borel set (*cf.*, *e.g.*, [Hal50]).

[6]A set of reals X has the **property of Baire** if $X \triangle O$ is meager for some open set O, where the **symmetric difference** $A \triangle B$ of two sets A and B is the set $(A \backslash B) \cup (B \backslash A)$, where $A \backslash B = \{x \in A : x \notin B\}$. A set of reals X has the **perfect set property** if it is countable or contains a perfect set (an uncountable closed set without isolated points). A set of reals X is **Lebesgue measurable** if there is a Borel set B such that $X \triangle B$ is a subset of a Borel measure 0 set. *Cf.* [Oxt80].

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Question 43 of the Scottish Book, posed by Mazur, asks about games where two players alternately select the members of a shrinking sequence of intervals of real numbers, with the first player the winner if the intersection of the sequence intersects a set given in advance. Banach posted an answer in 1935, showing that such games are determined if and only if the given set is either meager (in which case the second player wins) or comeager relative to some interval (in which case the first player wins). The determinacy of the restriction of this game to each interval implies then that the given set has the Baire property (*cf.* [Oxt80, pp. 27–30] and [Kan03, pp. 373–374]). The game has come to be known as the Banach-Mazur game. Using an enumeration of the rationals, one can code intervals with rational endpoints with integers, getting a game on integers.

Morton Davis [Dav64] studied a game, suggested by Lester Dubins, where the first player plays arbitrarily long finite strings of 0's and 1's and the second player plays individual 0's and 1's, with the payoff set a subset of the set of infinite binary sequences as before. Davis proved that the first player has a winning strategy in such a game if and only if the payoff set contains a perfect set, and the second player has a winning strategy if and only if the payoff set is finite or countably infinite. The determinacy of all such games then implies that every uncountable set of reals contains a perfect set (asymmetric games of this type can be coded by integer games of perfect information). It follows that under AD there is no set of reals whose cardinality falls strictly between $\aleph_0$ and $2^{\aleph_0}$.[7]

Mycielski and Świerczkowski showed that the determinacy of certain integer games of perfect information implies that every subset of the real line is Lebesgue measurable [MŚ64]. Simpler proofs of this fact were later given by Leo Harrington (*cf.* [Kan03, pp. 375–377]) and Martin [Mar03].

By way of contrast, an argument of Vitali [Vit05] shows that under ZFC there are sets of reals which are not Lebesgue measurable. Banach and Tarski (*cf.* [BT24]; *cf.* also [Wag93] for a modern exposition), building on work of Hausdorff [Hau14], showed that under ZFC the unit ball can be partitioned into five pieces which can be rearranged to make two copies of the same sphere, again violating Lebesgue measurability as well as physical intuition. As with the undetermined game given by Gale and Stewart, the constructions of Vitali and Banach-Tarski use the Axiom of Choice and do not give definable examples of nonmeasurable sets. Via the Mycielski-Świerczkowski theorem, determinacy results would rule out the existence of definable examples, for various notions of definability.

2.2. Definability. As discussed above, ZFC implies that open sets are determined, and implies also that there exists a nondetermined set. The study of determinacy merges naturally with the study of sets of reals in terms of their

[7] *I.e.*, for every set X, if there exist injections $f : \omega \to X$ and $g : X \to 2^{\omega}$, then either X is countable or there exists a bijection between X and 2^{ω}.

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definability (*i.e.*, descriptive set theory), which can be taken as a measure of their complexity. In this section we briefly introduce some important definability classes for sets of reals. Standard references include [Mos80, Kec95]. While we do mention some important results in this section, much of the section can be skipped on a first reading and used for later reference.

A **Polish space** is a topological space which is separable and completely metrizable. Common examples include the integers ω, the reals $\mathbb{R}$, the open interval $(0, 1)$, the Baire space ${}^{\omega}\omega$, the Cantor space ${}^{\omega}2$ and their finite and countable products. Uncountable Polish spaces without isolated points are a natural setting for studying definable sets of reals. For the most part we will concentrate on the Baire space and its finite powers.

Following notation introduced by Addison [Add58B],[8] open subsets of a Polish space are called $\utilde{\mathbf{\Sigma}}^0_1$, complements of $\utilde{\mathbf{\Sigma}}^0_n$ sets are $\utilde{\mathbf{\Pi}}^0_n$, and countable unions of $\utilde{\mathbf{\Pi}}^0_n$ sets are $\utilde{\mathbf{\Sigma}}^0_{n+1}$. More generally, given a positive $\alpha < \omega_1$, $\utilde{\mathbf{\Sigma}}^0_\alpha$ consists of all countable unions of members of $\bigcup_{\beta<\alpha} \utilde{\mathbf{\Pi}}^0_\beta$, and $\utilde{\mathbf{\Pi}}^0_\alpha$ consists of all complements of members of $\utilde{\mathbf{\Sigma}}^0_\alpha$. The Borel sets are the members of $\bigcup_{\alpha<\omega_1} \utilde{\mathbf{\Sigma}}^0_\alpha$.

A **pointclass** is a collection of subsets of Polish spaces. Given a pointclass $\Gamma \subseteq \wp({}^{\omega}\omega)$, we let $\mathrm{Det}(\Gamma)$ and Γ**-determinacy** each denote the statement that $\mathrm{G}_\omega(A)$ is determined for all $A \in \Gamma$. Philip Wolfe proved $\utilde{\mathbf{\Sigma}}^0_2$-determinacy in ZFC[Wol55]. Davis followed by proving $\utilde{\mathbf{\Pi}}^0_3$-determinacy [Dav64]. Jeffrey Paris would prove $\utilde{\mathbf{\Sigma}}^0_4$-determinacy [Par72]. However, this result was proved after Martin had used a measurable cardinal to prove analytic determinacy (*cf.* § 5.2).

Continuous images of $\utilde{\mathbf{\Pi}}^0_1$ sets are said to be $\utilde{\mathbf{\Sigma}}^1_1$, complements of $\utilde{\mathbf{\Sigma}}^1_n$ sets are $\utilde{\mathbf{\Pi}}^1_n$, and continuous images of $\utilde{\mathbf{\Pi}}^1_n$ sets are $\utilde{\mathbf{\Sigma}}^1_{n+1}$. For each $i \in \{0, 1\}$ and $n \in \omega$, the pointclass $\utilde{\mathbf{\Delta}}^i_n$ is the intersection of $\utilde{\mathbf{\Sigma}}^i_n$ and $\utilde{\mathbf{\Pi}}^i_n$. The **boldface projective pointclasses** are the sets $\utilde{\mathbf{\Sigma}}^1_n$, $\utilde{\mathbf{\Pi}}^1_n$, and $\utilde{\mathbf{\Delta}}^1_n$ for positive $n \in \omega$. These classes were implicit in work of Lebesgue as early as [Leb18]. They were made explicit in independent work by Nikolai Luzin [Luz25C, Luz25B, Luz25A] and Wacław Sierpiński [Sie25]. The notion of a boldface pointclass in general (*i.e.*, possibly non-projective) is used in various ways in the literature. We will say that a pointclass Γ is **boldface** (or **closed under continuous preimages** or **continuously closed**) if $f^{-1}[A] \in \Gamma$ for all $A \in \Gamma$ and all continuous functions f between Polish spaces (where A is a subset of the codomain). The classes $\utilde{\mathbf{\Sigma}}^0_\alpha$, $\utilde{\mathbf{\Pi}}^0_\alpha$, $\utilde{\mathbf{\Delta}}^0_\alpha$ are also boldface in this sense.

The pointclass $\utilde{\mathbf{\Sigma}}^1_1$ is also known as the class of **analytic sets**, and was given an independent characterization by Mikhail Suslin [Sus17]: A set of reals A is analytic if and only if there exists a family of closed sets D_s (for each finite

[8]The papers [Add58B] and [Add58A] appear in the same volume of ***Fundamenta Mathematicae***. The front page of the volume gives the date 1958–1959. The individual papers have the dates 1958 and 1959 on them, respectively.

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