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DISCRETE HARMONIC ANALYSIS

This self-contained book introduces readers to discrete harmonic analysis with an emphasis on the Discrete Fourier Transform and the Fast Fourier Transform on finite groups and finite fields, as well as their noncommutative versions. It also features applications to number theory, graph theory, and representation theory of finite groups. Beginning with elementary material on algebra and number theory, the book then delves into advanced topics from the frontiers of current research, including spectral analysis of the DFT, spectral graph theory and expanders, representation theory of finite groups and multiplicity-free triples, Tao's uncertainty principle for cyclic groups, harmonic analysis on GL(2,F_q), and applications of the Heisenberg group to DFT and FFT. With numerous examples, figures, and more than 160 exercises to aid understanding, this book will be a valuable reference for graduate students and researchers in mathematics, engineering, and computer science.

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Discrete Harmonic Analysis

Representations, Number Theory, Expanders, and the Fourier Transform

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University Printing House, Cambridge CB2 8BS, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

314–321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre, New Delhi - 110025, India

79 Anson Road, #06-04/06, Singapore 079906

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www.cambridge.org
Information on this title: www.cambridge.org/9781107182332
DOI: 10.1017/9781316856383

First published 2018

Printed in the United States of America by Sheridan Books, Inc.

A catalogue record for this publication is available from the British Library.

Library of Congress Cataloging-in-Publication Data
Names: Ceccherini-Silberstein, Tullio, author. | Scarabotti, Fabio, author. | Tolli, Filippo, 1968– author.
Title: Discrete harmonic analysis : representations, number theory, expanders, and the fourier transform / Tullio Ceccherini-Silberstein (Universitaa degli Studi del Sannio, Benevento, Italy), Fabio Scarabotti (Universitaa degli Studi di Roma 'La Sapienza', Italy), Filippo Tolli (Universitaa Roma Tre, Italy).
Other titles: Harmonic analysis
Description: Cambridge : Cambridge University Press, 2018. | Series: Cambridge studies in advanced mathematics | Includes bibliographical references and index.
Identifiers: LCCN 2017057902 | ISBN 9781107182332 (hardback : alk. paper)
Subjects: LCSH: Harmonic analysis. | Fourier transformations. | Finite groups. | Finite fields (Algebra)
Classification: LCC QA403 .C4285 2018 | DDC 515/.2433 – dc23 LC record available at https://lccn.loc.gov/2017057902

ISBN 978-1-107-18233-2 Hardback

Cambridge University Press
978-1-107-18233-2 — Discrete Harmonic Analysis
Tullio Ceccherini-Silberstein , Fabio Scarabotti , Filippo Tolli
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To the memory of my six (sic!) grandparents:
Tullio Levi-Civita, Padua 29.3.1873 – Rome 29.12.1941
Libera Trevisani Levi-Civita, Verona 17.5.1890 – Rome 11.12.1973
Riccardo Vittorio **Ceccherini** Stame, Rome 7.6.1903 – Rome 16.10.1991
Piera Paoletti Ceccherini, Florence 9.8.1896 – Rome 8.5.1973
Walter **Silberstein**, Vienna 14.7.1911 – Auschwitz 11.7.1944
Edith Hahn Silberstein, Vienna 7.2.1911 – Auschwitz 23.5.1944

To my parents, Cristina, Nadiya, and Virginia

To the memory of my father, and to my mother

Cambridge University Press
978-1-107-18233-2 — Discrete Harmonic Analysis
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Contents

Preface

The aim of the present monograph is to introduce the reader to some central topics in discrete harmonic analysis, namely character theory of finite Abelian groups, (additive and multiplicative) character theory of finite fields, graphs and expanders, and representation theory of finite (possibly not Abelian) groups, including spherical functions, associated Fourier transforms, and spectral analysis of invariant operators. An important transversal topic, which is present in several sections of the book, is constituted by tensor products, which are developed for matrices, graphs, and representations.

We have written the book to be as self-contained as possible: it only requires some elementary notions in linear algebra (including the spectral theorem and its applications), abstract algebra (first rudiments in the theory of finite groups and rings), and elementary number theory.

First of all, we study in detail the structure of finite Abelian groups and their automorphisms. We then introduce the corresponding character theory leading to a complete analysis of the Fourier transform, focusing on the connections with number theory. For instance, we deduce Gauss law of quadratic reciprocity from the spectral analysis of the Discrete Fourier Transform. Actually, characters of finite Abelian groups will appear also, as a fundamental tool in the proof of several deep results, in subsequent chapters, constituting this way the central topic and common thread of the whole book.

We also present Dirichlet's theorem on primes in arithmetic progressions, which is based on the character theory of finite Abelian groups as well as Tao's uncertainty principle for (finite) cyclic groups [157].

Our treatment also includes an exposition of the Fast Fourier Transform, focusing on the theoretical aspects related to its expressions in terms of factorizations and tensor products. This part of the monograph is inspired, at least partially, by the important work of Auslander and Tolimieri [15] and the papers by Davio [50] and Rose [130]. The book by Stein and Shakarchi [150] has been a fundamental source for our treatment of Dirichlet's theorem as well as for the first section of the chapter on the Fast Fourier Transform.

The second part of the book constitutes a self-contained introduction to the basic algebraic theory of finite fields and their characters. This includes, on the one hand, a complete study of the automorphisms, norms, traces, and quadratic extensions of finite fields and, on the other hand, additive characters and multiplicative characters and several associated sums (trigonometric and Gaussian) and the Fast Fourier Transform over finite fields. One of the main goals is to present the generalized Kloosterman sums from Piatetski-Shapiro's monograph [123], which will play a fundamental role in Chapter 14 on the representation theory of $\mathrm{GL}(2, \mathbb{F}_q)$. We also introduce the reader to the study, initiated by André Weil [165], of the number of solutions of equations over finite fields and present the Hasse-Davenport identity [48], which relates the Gauss sums over a finite field and those over a finite extension.

The third part is devoted to harmonic analysis on finite graphs and several constructions such as the replacement product and the zig-zag product. The central themes are expanders and Ramanujan graphs. We present the basic theorems of Alon-Milman and Dodziuk, and of Alon-Boppana-Serre, on the isoperimetric constant and the spectral gap of a (finite, undirected, connected) regular graph, and their connections. We discuss a few examples with explicit computations showing optimality of the bounds given by the above theorems. We then give the basic definitions of expanders and describe three fundamental constructions due to Margulis, to Alon, Schwartz, and Shapira (based on the replacement product), and to Reingold, Vadhan, and Wigderson (based on the zig-zag product). In these constructions, the harmonic analysis on finite Abelian groups and finite fields we developed in the previous parts plays a crucial role. The presentation is inspired by the monographs by Terras [159], Lubotzky [99], and by Davidoff-Sarnak-Valette [49], as well as by the papers by Hoory-Linial-Wigderson [74], Alon-Schwartz-Shapira [10], and Alon-Lubotzky-Wigderson [8].

The final part of the present monograph is devoted to the representation theory of finite groups with emphasis on induced representations and Mackey theory. This includes a complete description of the irreducible representations of the affine groups and Heisenberg groups with coefficients in both the finite field $\mathbb{F}_q$ and the ring $\mathbb{Z}/n\mathbb{Z}$. Moreover, both the Discrete Fourier Transform and the Fast Fourier Transform are revisited, following Auslander-Tolimieri [15] and Schulte [142], in terms of two different realizations of a particular representation of the Heisenberg group. In Chapter 13 we develop, with a complete and original treatment, the basic theory of multiplicity-free triples, their associated spherical functions, and (commutative) Hecke algebras. This is a subject that has not yet received the attention it deserves. As far as we know, this notion is just mentioned in some exercises in Macdonald's book [105]. The classical theory of finite Gelfand pairs, which constitutes a particular yet fundamental case, was essentially covered in our first monograph [29]. The exposition culminates with a complete treatment of the representation theory of $GL(2, \mathbb{F}_q)$, along the lines developed by Piatetski-Shapiro [123]: our approach, via multiplicity-free triples, constitutes our original contribution to the theory.

All this said, one can use this monograph as a textbook for at least four different courses on:

(i) **Finite Abelian groups, the DFT, and the FFT** (the structure of finite Abelian groups, their character theory, and the Fourier transforms): Sections 1.1, 1.2, and 1.3, and Chapters 2, 4, and 5. The remaining sections in Chapter 1 as well as Chapter 3 are optional.

(ii) **Finite commutative harmonic analysis** (the structure of finite Abelian groups, their character theory, and the Fourier transforms; Dirichlet's theorem; finite fields and their characters): Sections 1.1, 1.2, and 1.3, and Chapters 2, 3, 4, 6, and 7.

(iii) **Graph theory** (a brief introduction to finite graphs, various notions of graph products, spectral theory, and expanders): Sections 1.1, 1.2, 1.3, 2.1, 2.2, 2.3, and 2.4, and Chapters 8 and 9 (omitting, if necessary, the parts involving character theory of finite fields).

(iv) **Finite harmonic analysis** (representation theory of finite groups: from the basics to $GL(2, \mathbb{F}_q)$): Sections 1.1, 1.2, and 1.3, Chapters 2, 4, and 6, Sections 7.1, 7.2, 7.3, and 7.4, and the whole of Part IV (Section 12.5, Chapter 13, and Sections 14.7 and 14.8 may be omitted).

We thank Alfredo Donno for interesting discussions as well as for helping us with some figures. We also express our deep gratitude to Sam Harrison, Kaitlin Leach, Clare Dennison, Adam Kratoska, and Mark Fox from Cambridge University Press as well as the project manager Vijay Kumar Bhatia and the copyeditor Sara Barnes, for their constant encouragement and most precious help at all stages of the editing process.

Roma, 31 July 2017 TCS, FS, and FT