Cambridge University Press
978-1-107-18193-9 — Lectures on Infinitary Model Theory
David Marker
Excerpt
More Information

Chapter 1

INFINITARY LANGUAGES

Let τ be a vocabulary, i.e., a set of function symbols, relation symbols and constant symbols. In the logic $\mathcal{L}_{\infty,\omega}(\tau)$ we build formulas using the symbols from τ, equality, Boolean connectives $\neg$, $\bigwedge$ and $\bigvee$, quantifiers $\forall$ and $\exists$, variables $\{v_\alpha : \alpha \text{ an ordinal}\}$.[1]

- terms and atomic formulas are defined as in first order logic;
- if ϕ is a formula, then so is $\neg\phi$;
- if X is a set of formulas, then so are

$$\bigvee_{\phi\in X} \phi \text{ and } \bigwedge_{\phi\in X} \phi;$$

- if ϕ is a formula, so are $\exists v_\alpha\, \phi$ and $\forall v_\alpha\, \phi$.

We extend the usual definition of satisfaction by saying

$$\mathcal{M} \models \bigvee_{\phi\in X} \phi \text{ if and only if } \mathcal{M} \models \phi \text{ for some } \phi \in X$$

and

$$\mathcal{M} \models \bigwedge_{\phi\in X} \phi \text{ if and only if } \mathcal{M} \models \phi \text{ for all } \phi \in X.$$

For notational simplicity, we use the symbols $\wedge$ and $\vee$ as abbreviations for binary conjunctions and disjunctions. Similarly we will use the abreviations $\rightarrow$ and $\leftrightarrow$ when helpful.

We can inductively define the notions of *free variable*, *subformula*, *sentence*, *theory* and *satisfiability* in the usual ways.

Exercise 1.0.1. Suppose ϕ is an $\mathcal{L}_{\infty,\omega}$-sentence and ψ is a subformula of ϕ. Prove that ψ has only finitely many free variables.

Let κ be an infinite cardinal. In the logic $\mathcal{L}_{\kappa,\omega}(\tau)$ we form formulas in a similar way but we only use variables $\{v_\alpha : \alpha < \kappa\}$ and restrict infinite conjunctions and disjunctions to $\bigvee_{\phi\in X} \phi$ and $\bigwedge_{\phi\in X} \phi$ where $|X| < \kappa$. Thus $\mathcal{L}_{\omega,\omega}$ is just the usual first order logic. Throughout these notes we be focusing

[1] When no confusion arises we omit the τ and write $\mathcal{L}_{\infty,\omega}$ instead of $\mathcal{L}_{\infty,\omega}(\tau)$.

primarily on $\mathcal{L}_{\omega_1,\omega}$, the logic where we allow countable conjunctions and disjunctions.[2]

EXERCISE 1.0.2. Show that if κ is a regular cardinal and ϕ is a sentence of $\mathcal{L}_{\kappa,\omega}$, then ϕ has fewer than κ subformulas. Show that this fails for singular cardinals. This is one reason it is customary to restrict attention to $\mathcal{L}_{\kappa,\omega}$ for κ a regular cardinal.

DEFINITION 1.0.3. We say $\mathcal{M} \equiv_{\infty,\omega} \mathcal{N}$ if

$$\mathcal{M} \models \phi \text{ if and only if } \mathcal{N} \models \phi$$

for all $\mathcal{L}_{\infty,\omega}$ sentences ϕ. The notion $\mathcal{M} \equiv_{\kappa,\omega} \mathcal{N}$, is defined analogously.

EXERCISE 1.0.4. Show that if $\mathcal{M} \cong \mathcal{N}$, then $\mathcal{M} \equiv_{\infty,\omega} \mathcal{N}$.

When studying the model theory of infinitary logics there is one fundamental and inescapable fact:

The compactness theorem fails for infinitary languages.

EXERCISE 1.0.5. Let τ be the vocabulary with constant symbols $d, c_0, c_1, \ldots$ and let Γ be the set of sentences

$$\{d \neq c_i : i \in \omega\} \cup \left\{\forall v \bigvee_{i \in \omega} v = c_i\right\}.$$

Show that every finite subset of Γ is satisfiable, but Γ is not satisfiable. Thus the Compactness Theorem fails for $\mathcal{L}_{\infty,\omega}$ and even $\mathcal{L}_{\omega_1,\omega}$.

The failure of compactness will lead to many new phenomena and force us to find new approaches and develop new tools.[3]

EXERCISE 1.0.6. (a) Give an example of structures $\mathcal{M}_0, \mathcal{M}_1, \ldots$ and $\phi \in \mathcal{L}_{\omega_1,\omega}$ such that $\mathcal{M}_i \models \phi$ for all i, but if $\mathcal{U}$ is a nonprincipal ultrafilter on ω then $\prod M_i/\mathcal{U} \models \neg\phi$.

(b) Show that if $\mathcal{U}$ is a σ-complete ultrafilter on I, then

$$\prod_{i \in I} \mathcal{M}_i/\mathcal{U} \models \phi \Leftrightarrow \{i \in I : \mathcal{M}_i \models \phi\} \in \mathcal{U}$$

for $\phi \in \mathcal{L}_{\omega_1,\omega}$.

[2] Logically it would make sense to call this logic $\mathcal{L}_{\aleph_1,\aleph_0}$ but we will follow the historical precedent and refer to it as $\mathcal{L}_{\omega_1,\omega}$.

[3] Though we will not focus on it, another fruitful approach is to look for more general forms of the Compactness Theorem that hold in particular settings. One example of this is Barwise Compactness for countable admissible fragments. We will see an avatar of these results in Theorem 11.2.2 and give a quick treatment of the general result in Appendix B, but the interested reader should consult [8].

Another important example is studying compactness in languages $\mathcal{L}_{\kappa,\kappa}$ where κ is a large cardinal. See [31] or [30] for further information.

If compactness fails, why then do we study the model theory of infinitary languages? One reason is that we get new insights about first order model theory. But the simplest answer is that there are many natural classes that are axiomatized by $\mathcal{L}_{\omega_1,\omega}$-sentences.

Exercise 1.0.7. Show that the following classes are $\mathcal{L}_{\omega_1,\omega}$-axiomatizable for appropriate choices τ.

(a) torsion abelian groups;
(b) finitely generated groups;
(c) non-finitely generated groups;
(d) linear orders isomorphic to $(\mathbb{Z}, <)$;
(e) archimedean fields;
(f) connected graphs;
(g) recursively saturated models of PA;
(h) ω-models of ZFC (i.e., models of ZFC where the integers are standard);
(i) models of T omitting p, where T is a first order theory and p is a type.

Exercise 1.0.8. Show by compactness that none on these classes is axiomatizable in first order logic.

Examples (ii), (iv) and (v) immediately show the failure of the Upward Löwenheim–Skolem Theorem in infinitary languages. There are no uncountable linear orders isomorphic to $(\mathbb{Z}, <)$ and any archimedean ordered field is isomorphic to a subfield of the real numbers and hence has cardinality at most $2^{\aleph_0}$. We give another example in Exercise 1.1.13.

Exercise 1.0.9. Show by induction that for ordinal α there is an $\mathcal{L}_{\infty,\omega}$-sentence Φ_α describing $(\alpha, <)$ up to isomorphism.

Exercise 1.0.10. Let κ be a regular cardinal. Show there are $\alpha, \beta < (2^{2^\kappa})^+$ such that $(\alpha, <) \equiv_{\kappa,\omega} (\beta, <)$.

Taken together these two exercises give examples of structures $\mathcal{M}, \mathcal{N}$ with $\mathcal{M} \equiv_{\kappa,\omega} \mathcal{N}$ but $\mathcal{M} \not\equiv_{\infty,\omega} \mathcal{N}$ for all κ.[4]

We will also, from time to time, look at *pseudoelementary classes* which are simply reducts of elementary classes.

Definition 1.0.11. We say that a class $\mathcal{K}$ of τ-structures is a PC$_{\omega_1,\omega}$-*class* if there is a vocabulary $\tau^* \supseteq \tau$ and $\phi \in \mathcal{L}_{\omega_1,\omega}(\tau^*)$ such that

$$\mathcal{K} = \{\mathcal{M} : \text{ there is a } \tau\text{-structure } \mathcal{M}^* \text{ expanding } \mathcal{M} \text{ with } \mathcal{M}^* \models \phi\}$$

i.e., $\mathcal{K}$ is the class of τ-reducts of models of ϕ.

Similarly we say that $\mathcal{K}$ is a PC-*class* if ϕ is a first order sentence and $\mathcal{K}$ is a PC$_\delta$-*class* if ϕ is a first order theory.

[4] For a specific example: suppose $\kappa < \lambda$ are uncountable cardinals. Show that $\kappa \equiv_{\omega_1,\omega} \lambda$, but $\kappa \not\equiv_{\infty,\omega} \lambda$. See [43] for details.

EXERCISE 1.0.12. Show that the following classes are $\mathrm{PC}_{\omega_1,\omega}$.
(a) orderable groups;
(b) free groups;
(c) 1-trainsitive linear orders, i.e., linear orders where for any a, b there is an order automorphism taking a to b;
(d) fields that are pure transcendental extensions of $\mathbb{Q}$;
(e) incomplete ordered fields;
(f) ordered fields with an integer part (i.e., an ordered subring with no element between 0 and 1 such that every element of the field is within distance at most one of some element of the subring).

EXERCISE 1.0.13 (Silver). Let $\tau = \{U\}$ where U is unary and let $\mathcal{K} = \{\mathcal{M} : |\mathcal{M}| \leq 2^{|U(\mathcal{M})|} \wedge |\mathcal{M} \setminus U(\mathcal{M})| = |\mathcal{M}|\}$.
(a) Show that $\mathcal{K}$ is PC-class.
(b) Show that $\mathcal{K}$ is κ-categorical if and only if $\kappa = \beth_\alpha$ for some limit ordinal α.
This example shows that the straightforward generalization of Morley's Categoricity Theorem to PC-classes fails

EXERCISE 1.0.14. Let $\mathcal{K}$ be a $\mathrm{PC}_{\omega_1,\omega}$-class. Show that

$$\mathcal{K}_0 = \{\mathcal{M} \in \mathcal{K} : \mathcal{M} \text{ countable}\}$$

is also a $\mathrm{PC}_{\omega_1,\omega}$-class.

1.1. Fragments and Downward Löwenheim–Skolem

We will prove a useful and natural version of the Downward Löwenheim–Skolem Theorem. The next exercise shows that even here we will need to be careful.

EXERCISE 1.1.1. Give an example of a countable vocabulary τ and an $\mathcal{L}_{\omega_1,\omega}$-theory T such that every model of T has cardinality at least $2^{\aleph_0}$.

We will often restrict our attention to subcollections of the set of all $\mathcal{L}_{\omega_1,\omega}$-formulas. We will look at collections of formulas with some natural closure properties. One of these will be formal negation an operation that shows how we inductively move a negation inside a quantifier or a Boolean operation. Closure under formal negation isn't needed in this proof, but will be used in later arguments.

DEFINITION 1.1.2. For each formula ϕ we define $\sim\phi$, a *formal negation* of ϕ as follows:
(i) for ϕ atomic, $\sim\phi$ is $\neg\phi$;
(ii) $\sim(\neg\phi)$ is ϕ;
(iii) $\sim\bigwedge_{\phi\in X}\phi$ is $\bigvee_{\phi\in X}\sim\phi$ and $\sim\bigvee_{\phi\in X}\phi$ is $\bigwedge_{\phi\in X}\sim\phi$;
(iv) $\sim\exists v\phi$ is $\forall v\sim\phi$ and $\sim\forall v\phi$ is $\exists v\sim\phi$.

Exercise 1.1.3. Show by induction that

$$\mathcal{M} \models \neg\phi \text{ if and only if } \mathcal{M} \models {\sim}\phi$$

for all $\phi \in \mathcal{L}_{\infty,\omega}$.

Definition 1.1.4. We say that a set of $\mathcal{L}_{\infty,\omega}$-formulas $\mathbb{A}$ is a *fragment* if there is an infinite set of variables V such that if $\phi \in \mathbb{A}$, then all variables occurring in ϕ are in V and A satisfies the following closure properties:

(i) all atomic formulas using only the constant symbols of τ and variables from V are in $\mathbb{A}$;
(ii) if $\phi \in \mathbb{A}$ and ψ is a subformula of ϕ, then $\psi \in \mathbb{A}$;
(iii) If $\phi \in \mathbb{A}$, v is free in ϕ, and t is a term where every variable is in V, then the formula obtained by substituting t into all free occurrences of v is in $\mathbb{A}$;
(iv) $\mathbb{A}$ is closed under $\sim$;
(v) $\mathbb{A}$ is closed under $\neg$, $\wedge$, $\vee$, $\exists v$, and $\forall v$ for $v \in V$.

Exercise 1.1.5. (a) Suppose κ is regular (in particular this holds for $\mathcal{L}_{\omega_1,\omega}$). Prove that if T is a set of $\mathcal{L}_{\kappa,\omega}$-sentences with $|T| < \kappa$, then there is $\mathbb{A}$ a fragment of $\mathcal{L}_{\kappa,\omega}$ such that $T \subseteq \mathbb{A}$ and $|\mathbb{A}| < \kappa$.
(b) Show that there is a smallest such fragment.

Exercise 1.1.6. Let T and $\mathbb{A}$ be as above. Show that every formula in $\mathbb{A}$ has only finitely many free variables. Give an example showing that even though are only finitely many free variables, there may be infinitely many bound occurrences.

Exercise 1.1.7. We say that a formula is in *negation normal form* if the $\neg$ only occurs applied to atomic formulas. Show by induction that for any fragment $\mathbb{A}$ and any $\mathcal{L}_{\infty,\omega}$-formula $\phi \in \mathbb{A}$, there is $\psi \in \mathbb{A}$ such that ψ is equivalent to ϕ and ψ is in negation normal form.

Conjunctive and disjunctive normal form does not work as well.

Exercise 1.1.8. Let $\tau = \{P_{i,j} : i, j \in \omega\} \cup \{c\}$ where each $P_{i,j}$ is a unary predicate and c is a constant symbol. Consider the $\mathcal{L}_{\omega_1,\omega}$-sentence

$$\bigwedge_{i\in\omega}\bigwedge_{j\neq k}(P_{i,j}(c) \rightarrow \neg P_{i,k}(c)) \wedge \bigwedge_{i\in\omega}\bigvee_{j\in\omega} P_{i,j}(c).$$

Show that there is an equivalent sentence in disjunctive normal form in $\mathcal{L}_{(2^{\aleph_0})^+,\omega}$. (See also Exercise 3.1.6.)

We write $\mathcal{M} \equiv_{\mathbb{A}} \mathcal{N}$ and $\mathcal{M} \prec_A \mathcal{N}$ for elementary equivalence and elementary submodels with respect to formulas in $\mathbb{A}$ (where $\mathbb{A}$ could be $\mathcal{L}_{\omega_1,\omega}$ or $\mathcal{L}_{\infty,\omega}$).

Theorem 1.1.9 (Downward Löwenheim–Skolem). *Let $\mathbb{A}$ be a fragment of $\mathcal{L}_{\infty,\omega}$ such that any formula $\phi \in \mathbb{A}$ has at most finitely many free variables.*

Let $\mathcal{M}$ be a τ-structure with $X \subseteq \mathcal{M}$. There is $\mathcal{N} \preceq_{\mathbb{A}} \mathcal{M}$ with $X \subseteq \mathcal{N}$ and $|\mathcal{N}| \leq \max(|\mathbb{A}|, |X|)$.

In particular, if τ is countable and $\mathbb{A}$ is a countable fragment of $\mathcal{L}_{\omega_1,\omega}(\tau)$, then every τ-structure has a countable $\mathbb{A}$-elementary submodel.

The proof is a simple generalization of the proof in first order logic. It is outlined in the following Exercise.

EXERCISE 1.1.10. (a) Prove there is $\tau^* \supseteq \tau$ and $\mathbb{A}^* \supseteq \mathbb{A}$ and $\mathcal{M}^*$ an τ^*-expansion of $\mathcal{M}$ such that $|\tau^*|, |\mathbb{A}^*| = |\mathbb{A}|$ and for each $\mathbb{A}^*$-formula $\phi(\overline{v}, w)$ with free variables from $v_1, \ldots, v_n, w$, there is an n-ary function symbol $f_\phi \in \tau^*$ such that

$$\mathcal{M}^* \models \forall \overline{v}\, (\exists w\, \phi(\overline{v}, w) \to \phi(\overline{v}, f_\phi(\overline{v}))).$$

(b) Prove that if $\mathcal{N}$ is a τ^*-substructure of $\mathcal{M}^*$, then $\mathcal{N} \preceq_{\mathbb{A}^*} \mathcal{M}^*$.
(c) Prove that there is a τ^*-substructure $\mathcal{N}$ of $\mathcal{M}^*$ with $X \subseteq \mathcal{N}$ and $|\mathcal{N}| \leq \max(|\mathbb{A}^*|, |X|)$.

EXERCISE 1.1.11. For τ and $\mathbb{A}$ define the appropriate notion of *built-in Skolem functions*. Prove that for any τ and $\mathbb{A}$ and $\mathbb{A}$-theory T there are $\tau^* \supseteq \tau$, $\mathbb{A}^* \supseteq \mathbb{A}$ and $T^* \supseteq T$ with $|\mathbb{A}^*| = |T^*| = |\mathbb{A}|$ where T^* has built-in Skolem functions and any model of T has and expansion to a model of T^*.

EXERCISE 1.1.12. Let $\mathbb{A}$ be a fragment of $\mathcal{L}_{\infty,\omega}$ where every formula has finitely many free variables and let $\mathcal{K}$ be a $\mathrm{PC}_{\omega_1,\omega}$-class. Show that if $\mathcal{M} \in \mathcal{K}$ is a τ-structure and $X \subset \mathcal{M}$, then there is $\mathcal{N} \prec_{\mathbb{A}} \mathcal{M}$ such that $|\mathcal{N}| = \max(|X|, |\mathbb{A}|)$ and $\mathcal{N} \in \mathcal{K}$.

EXERCISE 1.1.13. As mentioned above, in first order logic, the Upward Löwenheim-Skolem is an easy consequence of Compactness. In infinitary logics it is generally false. Let $\tau = \{U, S, E, c_1, \ldots, c_n, \ldots\}$, where U and S are unary predicates, E is a binary predicate and each c_i is a constant. Let ϕ be the conjunction of

(i) $\forall x\, U(x) \leftrightarrow \neg S(x)$,
(ii) $\forall x \forall y\, (E(x, y) \to U(x) \wedge S(y))$,
(iii) $c_i \neq c_j$, for $i \neq j$,
(iv) $U(c_i)$ for all i,
(v) $\forall y \forall z\, ([S(y) \wedge S(z) \wedge \forall x\, ((E(x, y) \leftrightarrow E(x, z))] \to y = z)$,
(vi) $\forall x (U(x) \to \bigvee_{i=1}^{\infty} x = c_i)$.

Prove that every model of ϕ has size at most $2^{\aleph_0}$.

Elementary chains behave as they do in first order logic.

EXERCISE 1.1.14. Suppose $\mathbb{A}$ is a fragment of $\mathcal{L}_{\infty,\omega}$, $(I,<)$ is a linear order and $(\mathcal{M}_i : i \in I)$ is an elementary chain of τ-structures, i.e., $\mathcal{M}_i \preceq_{\mathbb{A}} \mathcal{M}_j$ for $i < j$. Let $\mathcal{M} = \bigcup \mathcal{M}_\alpha$. Then $\mathcal{M}_i \preceq_{\mathbb{A}} \mathcal{M}$ for all $i \in I$.

EXERCISE 1.1.15. Give an example showing that PC classes need not be preserved by elementary chains.

EXERCISE 1.1.16. Let τ be countable. Suppose $(\mathcal{M}_\alpha : \alpha < \omega_1)$ is a chain of countable τ-structures such that $\mathcal{M}_\beta = \bigcup_{\alpha<\beta} \mathcal{M}_\alpha$ for β a limit ordinal. Let $\mathcal{M} = \bigcup_{\alpha<\omega_1} \mathcal{M}_\alpha$. Suppose $\phi \in \mathcal{L}_{\omega_1,\omega}$ and $\mathcal{M} \models \phi$. Show that $\{\alpha : \mathcal{M}_\alpha \models \phi\}$ is closed unbounded.

1.2. $\mathcal{L}_{\omega_1,\omega}$ and omitting first order types

In Exercise 1.0.7 we noted that the class of models of a first order theory omitting a type is expressible in $\mathcal{L}_{\omega_1,\omega}$. The next result, due to Chang, shows that any class axiomatizable by an $\mathcal{L}_{\omega_1,\omega}$-sentence is the reduct of a class of models of a first order theory omitting a set of types.

THEOREM 1.2.1. *Let τ be a countable vocabulary and let T be a countable set of $\mathcal{L}_{\omega_1,\omega}$-sentences. There is a countable vocabulary $\tau^* \supseteq \tau$, a first order τ^*-theory T^* and a set of partial types Γ such that:*

(i) *if $\mathcal{M} \models T^*$ and $\mathcal{M}$ omits all types in Γ, then the τ-reduct of $\mathcal{M}$ is a model of T;*
(ii) *every model of T has an τ^*-expansion that is a model of T^* omitting all types in Γ.*

PROOF. Let $\mathbb{A}$ be the smallest fragment containing all sentences in T. We expand τ to τ^* so for each formula ϕ in $\mathbb{A}$ with free variables from $v_1, \ldots, v_n$ we have an n-ary relation symbol R_ϕ. T^* is formed by taking the following sentences:

(i) if ϕ is atomic add $\forall \overline{v}\,(R_\phi \leftrightarrow \phi)$;
(ii) if ϕ is $\neg\theta$ add $\forall \overline{v}\,(R_\phi \leftrightarrow \neg R_\theta)$;
(iii) if ϕ is $\bigwedge_{\theta\in X} \theta$ add $\forall \overline{v}\,(R_\phi \rightarrow R_\theta)$ for all $\theta \in X$ and let γ_ϕ be the type $\{\neg R_\phi\} \cup \{R_\theta : \theta \in X\}$;
(iv) if ϕ is $\bigvee_{\theta\in X} \theta$, add $\forall \overline{v}\, R_\theta \rightarrow R_\phi$ for all θ in X and let γ_ϕ be the type $\{R_\phi\} \cup \{\neg R_\theta : \theta \in X\}$;
(v) if ϕ is $\exists w\,\theta$, then add $\forall \overline{v}\,(R_\phi \leftrightarrow \exists w R_\theta)$;
(vi) if ϕ is $\forall w\,\theta$, then add $\forall \overline{v}\,(R_\phi \leftrightarrow \forall \overline{v} \forall w\, R_\theta)$;
(vii) for each sentence $\phi \in T$, add R_ϕ to T^*.

Let Γ be the collection of types γ_ϕ described above.[5]

[5] As described here for each sentence ϕ we have added a 0-ary predicate symbol. If you are not comfortable with this approach we could rephrase this by adding a single constant c. Then for each sentence ϕ we could add unary predicate R_ϕ and make assertions about $R_\phi(c)$.

EXERCISE 1.2.2. (a) Suppose $\mathcal{M} \models T^*$ and $\mathcal{M}$ omits every type in Γ. Prove that

$$\mathcal{M} \models \forall \overline{v}\, \phi \leftrightarrow R_\phi$$

for all $\phi \in \mathbb{A}$.

Conclude that the τ-reduct of a model of T^* is a model of T.

(b) Prove that every $\mathcal{M} \models T$ has an expansion that is a model of T^* omitting all types in Γ by interpreting R_ϕ as ϕ.

This completes the proof. ⊣

We will examine a refinement of this result in Theorem 2.2.18.