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978-1-107-18091-8 — Cycles and Social Choice
Thomas Schwartz
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Introduction

The mathematical theory of voting and social choice grew from a tiny seed of an example discovered over two centuries ago by the Marquis de Condorcet. Now called the Paradox of Voting, it shows that majority preferences can form a cycle: sometimes majorities prefer x to y, y to z, and z to x. That makes any choice from among x, y, and z unstable under majority rule: whatever may be chosen, some voters have the power to reject that choice in favor of one that they like more. Contemporary contributions to the theory still treat of cycles and instability, under majority rule and other regimes, along with all sorts of generalizations, variations, applications, implications, and interpretations. They form much of the deductive foundation of what we know and how we learn about election systems, legislative procedure, and constitutions, and to some extent private organizations, administrative processes, and exchange economies. Despite the precision of mathematics, however, or maybe because of it, the most famous contributions are more often celebrated than understood. Countless expositions and references to key findings suffer from gross error, yawing gaps, and meretricious formulation, often hidden behind needless notation.

I mean to retell the whole story of cycles and instability, their sources and consequences, as simply and soundly as possible, scrapping otiose apparatus (but without sacrificing rigor), correcting errors (though rarely naming errants), filling gaps (but sidestepping inconsequential variations and scholarly qualifications), and adding episodes never before told. Like a great flower, this story unfolds in multiple directions. Half of it is about the diverse sources of cycles. Much of that diversity comes from the cyclic relation itself: it is not always majority preference. Other relations of "social preference," of

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winners to losers in pairwise contests, can be cyclic too. The other half, no doubt the more interesting half to many readers, is about consequences. Cycles are not puzzles or maladies to be solved or cured but positive sources of knowledge about how and why things work when preferences, strategies, and procedures conspire to produce social or collective choices – choices attributable to all of certain actors but not to any one of them. Sometimes seen as a mark of incoherence, cycles are, in a way, the very opposite. By dint of their many and varied sources and consequences, they impart coherence to a host of diverse and apparently unconnected features of society.

In deciding what details to include and how much space to give them, I have favored hard results over blatherskite, simple results over more complicated ones to the same effect, intelligibility over impressiveness, English over needless notation, Euclidean over analytic geometry, recyclable forms of proof over single-use ones, novelty over orthodoxy, and truth over error (which I apologize for taxing your patience to correct). More important, I have favored procedure over preference. Chapter 2 is about preferential sources of cycles and instability and their absence, the ways in which different combinations of voter preferences induce or block cycles and instability. Most everything else in the first half is about procedural sources, about the sorts of procedure that allow cycles and instability and their refinements and consequences. I have emphasized procedure partly out of personal interest (I assume you prefer to read what I write about things I know most about), but partly too because a more thorough treatment of the preferential stuff would give too much space to results that look impressive, and are in technical ways, but that ultimately rest on false assumptions. More than once, and more than most authors on social-choice theory, I have invoked the admittedly unfashionable criterion of truth to separate acceptable from unacceptable assumptions, and with it sound from unsound arguments.

For readers who know something about the mathematical theory of voting and social choice, here is a selective preview. Chapter 1 begins

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with the Paradox of Voting itself, Condorcet's little example. There I disparage portentous interpretations and add some simple generalizations and variations, just enough to show, without any apparent mathematics, that cycles are not peculiar to majority rule or even voting – though along the way I explain the peculiar significance of majority rule and show how it can spawn much more than cycles. One of those variations is Sen's Liberal Paradox, which reveals cycles based on individual rights. It is usually illustrated with exotic or contentious examples, but I show how humdrum and ubiquitous it really is: it is exemplified by every economic exchange.

Chapter 2 addresses those preferential sources of cycles and their absence. It begins where twentieth-century research on the mathematics of voting and social choice itself began, with Duncan Black's condition of Single Peakedness, or one-dimensionality, and his famous median-stability theorem. In a world of two or more dimensions, stability gives way to near-certain instability, thanks chiefly to Charles Plott's Pairwise Symmetry theorem. That result is supposed to be mathematically challenging, but it follows almost trivially from Black's stability theorem. Alas, that and other spatial instability results have been over-interpreted: they show less than is sometimes alleged. A more revealing source of cycles and instability is issue packaging, as when votes are traded or draft laws are composed of simple measures that cannot pass separately. It is still true that a one-dimensional or single-peaked world would be free of cycles and instability, but widely cited empirical evidence of such a world, based on dimensional analysis of legislative votes, turns out to be spurious.

As a prelude to the procedural sources of cycles, I introduce Kenneth Arrow's celebrated Impossibility Theorem in Chapter 3, but only as a prelude. The theorem itself says nothing at all about cycles. It asserts a contradiction between a set of very mild background assumptions – one of them widely misinterpreted as rather restrictive – and a very strong transitivity assumption. That assumption does ban cycles, but it bans much more, and a ban on cycles alone

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is consistent with those background assumptions: they do not imply that cycles ever exist. I introduce Arrow anyway because Chapter 4 shows that we can add just a bit to those assumptions and get a cycle (or contradict a ban on cycles). Actually, there are several ways to do that, the first discovered by my 26-year-old self, but the one most often cited rests on a meretricious assumption, Positive Responsiveness, innocent looking but almost always false.

Another route to cycles bypasses Arrow and starts with Condorcet. Several direct generalizations of his example are on the books. Chapter 5 offers a sweeping generalization of them all, a simple sufficient condition for cycles that is demonstrably as general as possible, necessary as well as sufficient. After that I pause, in Chapter 6, to recast all those results in terms of a fixed set of feasible alternatives containing a top cycle – and one with several special properties.

Chapter 7 turns from sources to consequences, strategic ones first. One of them is strategic manipulability, the ability of voters to profit from misrepresenting their true preferences. Actually, we can generalize and assume a bit less than cycles, then use that result as a lemma to prove that manipulability is quite inescapable: any procedure for choosing among three or more alternatives must be manipulable unless it is purely dictatorial. Such is the Duggan–Schwartz Theorem. It differs from the older, widely misinterpreted Gibbard–Satterthwaite Theorem in dropping the generally false assumption of resoluteness.

Two more strategic consequences of cycles, also in Chapter 7, are about game solutions, specifically the core and the set of Nash equilibria. As framed in Chapter 6, social-choice procedures are tantamount to game forms, structures that become games when players' preferences are supplied. The core of a game is the set of outcomes that no coalition has the power and incentive to change. Its Nash equilibria are those outcomes that no single player has the power and incentive to change. A famous and obvious connection to cycles is that they make cores empty. So no social-choice procedure that allows cycles can be

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implemented, as we say, by the core: what the procedure produces is not, in general, the core of a game. You may be surprised to see that cycles also block implementation by the set of Nash equilibria.

Structural consequences of cycles are more numerous. Several have to do with the forms of legislative agendas, very broadly understood. In one sense, an agenda is a tree-like structure that tells us the order of pairwise votes and how later votes depend, if at all, on earlier votes. Assuming that voters act strategically, cycles make the final outcome depend on agenda structure of that sort, even when the set of alternatives on the agenda is held fixed. In another sense, the agenda is just the set of feasible outcomes. The final outcome depends on that set, of course, but how dependent it is – how sensitive to changes in that set – varies with the procedure being used. It is especially dependent when that procedure allows cycles. Yet a third kind of agenda structure consists of the combination or division of "questions," as when legislative items are assembled to form complex packages or divided into simpler components. As you have doubtless guessed, it is cycles that make the final outcome sensitive to such structural differences.

Another important structure is the division of government into constitutional components that must concur on policy and therefore can veto each other's acts. Two houses of a legislature are so related, as is the legislature as a whole, in many cases, to independent executives and courts. Offhand, that power looks like an asset, advantageous to its possessor and therefore a source of compromise and of checks and balances – but not when cycles are present. Then the veto can be downright disadvantageous, not as a thing to do but as a power to possess.

Finally, there are political parties. Often they are not imposed by law. So why are they there? And how exactly do they differ from other coalitions? The answers lie in cycles: without them, Chapter 8 concludes, there would be no parties.

Cycles are consequential in yet another way, apparently nugatory but ultimately constructive: by ruling out simple solutions to

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problems of prediction and prescription, of explanation and evaluation, cycles pose questions and help produce answers that fill gaps in knowledge. On the positive side, one of those questions is about solutions to social-choice processes, or cooperative games: how to predict outcomes in the face of empty cores. Another is how to explain observed stability. On the normative side, we find the oldest question occasioned by cycles, the one that exercised the Marquis himself and his great mathematical contemporary and countryman, Jean Charles de Borda: how to decide elections, or what is the ideal voting rule. Other questions are about the measurement of utility, or preference strength, the nature of welfare and its connection to social choice, the merits of conventional "rationality" requirements for individual as well as social choice, and our seemingly incurable conviction that we ought always to make best choices from sets of alternatives that have, ineluctably, been given us. If my explorations of those questions, in Chapters 9 and 10, do not always end with definite answers, it is because the questions themselves are not always definite enough to answer.

Throughout, I have minimized scholarship, or citations and surveys of related results. The chapter-by-chapter reviews under Background and Sources are meant to fill that gap.

I have written for four audiences. One is myself. Having thought about the Paradox of Voting on and off for many years, discovered many of the results reported here, been annoyed by repeated misstatements of important assumptions and findings, worked as a teacher to simplify arguments and recycle them as much as possible, and become ever more aware of cycles as a source of coherence, I naturally wished to set a lot of it down on paper to see what it all looked like. What I saw required refinement and inspired new findings.

Another intended audience is the community of social-choice specialists – or if they are not a real community, it may be because they could never agree on how to reach collective decisions. Their specific interests often differ from mine, but they can see immediately – and appreciate, I hope – where I have filled gaps, simplified proofs, found

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fault with familiar interpretations, and most of all extended knowledge. I expect some of them to challenge things I have said but hope some will feel challenged to apply and extend those things.

Then there are scholars and scientists who know something of my subject and talk about it from time to time, interpreting and applying its findings, but are not narrow specialists. I have tried to please them with a combination of succinctness, light technical demands, and applications to an array of independently interesting phenomena. At the risk of rudeness, I have also tried to correct them. For it is mostly they who, unwittingly, spread error, among themselves and to their own wider audiences.

The final audience consists of students but also of accomplished scholars and scientists who are merely aware of my subject and face a common conundrum: they wonder if that subject is worth the effort to learn about it, but the only way to find out is to learn about it. I have tried to help them by covering more content in less space with less apparatus but more attention to applications than any other book on the mathematical theory of voting and social choice. To gratify all four audiences – even myself, when I set out to assemble all these pieces about two years ago – I have striven to sprinkle the story of cycles and their sources and consequences with surprises.

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1 Condorcet's Two Discoveries

Yes, two, the first a prelude to the Paradox of Voting. Both come from his 1785 *Essai* and show, in different ways, how democratic electoral choices can flout the preferences of majorities. Sweeping generalizations of the Paradox appear in later chapters, but right away you will see that cycles are not at all peculiar to majority rule or even voting – but also that majority rule itself and the cycles it generates are peculiarly noteworthy.

1.1 THE REJECTION OF CONDORCET WINNERS

Suppose voters are divided into three minority factions, who rank three candidates in order of preference thus:

Liberals	Moderates	Conservatives
Libby	Maude	Connie
Maude	Libby and Connie	Maude
Connie	in either order	Libby

If Liberals are the largest faction and everyone votes for his favorite, Libby wins under Plurality Rule: she has the most votes. But a majority (Moderates and Conservatives) prefer Maude to Libby, and for that matter another majority prefer Maude to Connie. So Plurality Rule can reject a candidate preferred to all others by majorities. We call such a candidate the *Condorcet winner* – even when he loses.

The rejection of Condorcet winners happens with fair frequency when elections between two major parties give way to three-party contests. In the 1970 election for US Senator from New York, Conservative James Buckley won with a plurality, not a majority,

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defeating both the Democratic candidate and the Republican incumbent, Charles Goodell, who was the Condorcet winner: he occupied Maude's position. Then within a year, Socialist Salvador Allende received a 36 percent plurality for President of Chile, where the Condorcet winner was Christian Democrat Radomiro Tomic. Because Allende's total was less than 40 percent, the final choice was up to Congress, where the conservative Nationalists were willing to support Tomic. Alas, the Christian Democrats voted with the Socialists for Allende on the principle that the plurality favorite, not the Condorcet winner, is the true popular choice. In New York, Senator Buckley quickly restored two-party competition by joining the Republicans. In Chile, President Allende quickly broke his promise to the Christian Democrats to govern constitutionally. You know the rest.

This problem – if it is one – is not peculiar to Plurality Rule. The Double-Vote Rule, which requires a runoff when no candidate receives a majority of votes, might choose Maude. But maybe not: a runoff might pit Libby against Connie. A runoff would have bypassed Goodell and Tomic, each of whom had the fewest votes. Popular among social-choice theorists are the Borda and Approval Rules. The former chooses the candidate with the greatest Borda score, got by finding his rank from the bottom (0, 1, 2, etc.) in each voter's preference ordering and summing those ranks across all voters. Let five voters rank three candidates as follows:

	1	2	3	4	5	Borda scores:
2	x	x	x	y	y	$x = 6$
1	y	y	y	z	z	$y = 7$
0	z	z	z	x	x	$z = 2$

Then y has the greatest Borda score though x is the Condorcet winner. Like Plurality Rule, Approval Rule picks the candidate with the most votes, but it allows each voter to vote for (to "approve") more

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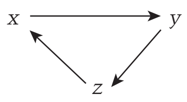
than one candidate. In that same example, if every voter votes for his first and second choices, y wins and the Condorcet winner is again rejected.

1.2 THE PARADOX OF VOTING

Once we have preferential ballots, ones that rank candidates in order of preference, we can spot and choose any Condorcet winner. But that does no good if there is no Condorcet winner to spot, as in Condorcet's second example, the Paradox of Voting:

1	2	3
x	y	z
y	z	x
z	x	y



Here a majority (voters 1 and 3) prefer x to y, another (1 and 2) y to z, and a third (2 and 3) z to x. As depicted, the relation of majority preference is *cyclic*. Not only is there no Condorcet winner (that can happen when two candidates are tied), but every possible choice is *unstable* under majority rule: whichever candidate is chosen, some majority prefers a different choice.

Cycles are not limited to three voters or candidates. Assume any number n of voters. So long as $3 \leq n \neq 4$, we can always divide them into three minority factions and assign them Condorcet's three preference orderings, one to each faction. Because each of the three is a minority, any two make a majority. So it is still true that majorities prefer x to y, y to z, and z to x. And instead of three candidates, assume *three or more alternatives*, however many and of whatever sort you please. Let them include x and y, but now let z be the *set* of all the rest. Then majorities still prefer x to y, but now y to every alternative in z and each of the latter to x – an all-inclusive cycle.

Our symmetric 3×3 picture might suggest that cycles make the social choice a matter of indifference: it matters not what is chosen. But consider another example:

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