

## Linear Algebra

*Linear Algebra* offers a unified treatment of both matrix-oriented and theoretical approaches to the course, which will be useful for classes with a mix of mathematics, physics, engineering, and computer science students. Major topics include singular value decomposition, the spectral theorem, linear systems of equations, vector spaces, linear maps, matrices, eigenvalues and eigenvectors, linear independence, bases, coordinates, dimension, matrix factorizations, inner products, norms, and determinants.

## CAMBRIDGE MATHEMATICAL TEXTBOOKS

Cambridge Mathematical Textbooks is a program of undergraduate and beginning graduate level textbooks for core courses, new courses, and interdisciplinary courses in pure and applied mathematics. These texts provide motivation with plenty of exercises of varying difficulty, interesting examples, modern applications, and unique approaches to the material.

### ADVISORY BOARD

John B. Conway, *George Washington University*  
Gregory F. Lawler, *University of Chicago*  
John M. Lee, *University of Washington*  
John Meier, *Lafayette College*  
Lawrence C. Washington, *University of Maryland, College Park*

A complete list of books in the series can be found at  
[www.cambridge.org/mathematics](http://www.cambridge.org/mathematics)

Recent titles include the following:

*Chance, Strategy, and Choice: An Introduction to the Mathematics of Games and Elections*, S. B. Smith  
*Set Theory: A First Course*, D. W. Cunningham  
*Chaotic Dynamics: Fractals, Tilings, and Substitutions*, G. R. Goodson  
*Introduction to Experimental Mathematics*, S. Eilers & R. Johansen  
*A Second Course in Linear Algebra*, S. R. Garcia & R. A. Horn  
*Exploring Mathematics: An Engaging Introduction to Proof*, J. Meier & D. Smith  
*A First Course in Analysis*, J. B. Conway  
*Introduction to Probability*, D. F. Anderson, T. Seppäläinen & B. Valkó  
*Linear Algebra*, E. S. Meckes & M. W. Meckes

# Linear Algebra

---

**ELIZABETH S. MECKES**

*Case Western Reserve University, Cleveland, OH, USA*

**MARK W. MECKES**

*Case Western Reserve University, Cleveland, OH, USA*



**CAMBRIDGE**  
UNIVERSITY PRESS

**CAMBRIDGE**  
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

314–321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre, New Delhi – 110025, India

79 Anson Road, #06–04/06, Singapore 079906

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning, and research at the highest international levels of excellence.

[www.cambridge.org](http://www.cambridge.org)

Information on this title: [www.cambridge.org/9781107177901](http://www.cambridge.org/9781107177901)

DOI: 10.1017/9781316823200

© Elizabeth S. Meckes and Mark W. Meckes 2018

This publication is in copyright. Subject to statutory exception and to the provisions of relevant collective licensing agreements, no reproduction of any part may take place without the written permission of Cambridge University Press.

First published 2018

Printed in the United States of America by Sheridan Books, Inc, June 2018

*A catalog record for this publication is available from the British Library.*

*Library of Congress Cataloging-in-Publication Data*

Names: Meckes, Elizabeth S., author. | Meckes, Mark W., author.

Title: Linear algebra / Elizabeth S. Meckes (Case Western Reserve University, Cleveland, OH, USA), Mark W. Meckes (Case Western Reserve University, Cleveland, OH, USA).

Description: Cambridge : Cambridge University Press, [2018] |

Includes bibliographical references and index.

Identifiers: LCCN 2017053812 | ISBN 9781107177901 (alk. paper)

Subjects: LCSH: Algebras, Linear—Textbooks.

Classification: LCC QA184.2 .M43 2018 | DDC 512/.5—dc23

LC record available at <https://lccn.loc.gov/2017053812>

ISBN 978-1-107-17790-1 Hardback

Cambridge University Press has no responsibility for the persistence or accuracy of URLs for external or third-party internet websites referred to in this publication and does not guarantee that any content on such websites is, or will remain, accurate or appropriate.

To Juliette and Peter

# Contents

---

<i>Preface</i>	<i>page</i> xiii
<i>To the Student</i>	xvii
<b>1 Linear Systems and Vector Spaces</b>	<b>1</b>
1.1 Linear Systems of Equations	1
Bread, Beer, and Barley	1
Linear Systems and Solutions	4
1.2 Gaussian Elimination	9
The Augmented Matrix of a Linear System	9
Row Operations	11
Does it Always Work?	14
Pivots and Existence and Uniqueness of Solutions	18
1.3 Vectors and the Geometry of Linear Systems	24
Vectors and Linear Combinations	24
The Vector Form of a Linear System	27
The Geometry of Linear Combinations	29
The Geometry of Solutions	33
1.4 Fields	39
General Fields	39
Arithmetic in Fields	42
Linear Systems over a Field	44
1.5 Vector Spaces	49
General Vector Spaces	49
Examples of Vector Spaces	53
Arithmetic in Vector Spaces	57
<b>2 Linear Maps and Matrices</b>	<b>63</b>
2.1 Linear Maps	63
Recognizing Sameness	63
Linear Maps in Geometry	65
Matrices as Linear Maps	67
Eigenvalues and Eigenvectors	69

	The Matrix–Vector Form of a Linear System	73
2.2	More on Linear Maps	78
	Isomorphism	78
	Properties of Linear Maps	80
	The Matrix of a Linear Map	83
	Some Linear Maps on Function and Sequence Spaces	86
2.3	Matrix Multiplication	90
	Definition of Matrix Multiplication	90
	Other Ways of Looking at Matrix Multiplication	93
	The Transpose	96
	Matrix Inverses	97
2.4	Row Operations and the LU Decomposition	102
	Row Operations and Matrix Multiplication	102
	Inverting Matrices via Row Operations	105
	The LU Decomposition	107
2.5	Range, Kernel, and Eigenspaces	114
	Range	115
	Kernel	118
	Eigenspaces	120
	Solution Spaces	123
2.6	Error-correcting Linear Codes	129
	Linear Codes	129
	Error-detecting Codes	130
	Error-correcting Codes	133
	The Hamming Code	134
<b>3</b>	<b>Linear Independence, Bases, and Coordinates</b>	<b>140</b>
3.1	Linear (In)dependence	140
	Redundancy	140
	Linear Independence	142
	The Linear Dependence Lemma	145
	Linear Independence of Eigenvectors	146
3.2	Bases	150
	Bases of Vector Spaces	150
	Properties of Bases	152
	Bases and Linear Maps	155
3.3	Dimension	162
	The Dimension of a Vector Space	163
	Dimension, Bases, and Subspaces	167
3.4	Rank and Nullity	172
	The Rank and Nullity of Maps and Matrices	172
	The Rank–Nullity Theorem	175

---

**Contents**

ix

Consequences of the Rank–Nullity Theorem	178
Linear Constraints	181
<b>3.5 Coordinates</b>	<b>185</b>
Coordinate Representations of Vectors	185
Matrix Representations of Linear Maps	187
Eigenvectors and Diagonalizability	191
Matrix Multiplication and Coordinates	193
<b>3.6 Change of Basis</b>	<b>199</b>
Change of Basis Matrices	199
Similarity and Diagonalizability	203
Invariants	206
<b>3.7 Triangularization</b>	<b>215</b>
Eigenvalues of Upper Triangular Matrices	215
Triangularization	218
<b>4 Inner Products</b>	<b>225</b>
<hr/>	
<b>4.1 Inner Products</b>	<b>225</b>
The Dot Product in $\mathbb{R}^n$	225
Inner Product Spaces	226
Orthogonality	229
More Examples of Inner Product Spaces	233
<b>4.2 Orthonormal Bases</b>	<b>239</b>
Orthonormality	239
Coordinates in Orthonormal Bases	241
The Gram–Schmidt Process	244
<b>4.3 Orthogonal Projections and Optimization</b>	<b>252</b>
Orthogonal Complements and Direct Sums	252
Orthogonal Projections	255
Linear Least Squares	259
Approximation of Functions	260
<b>4.4 Normed Spaces</b>	<b>266</b>
General Norms	267
The Operator Norm	269
<b>4.5 Isometries</b>	<b>276</b>
Preserving Lengths and Angles	276
Orthogonal and Unitary Matrices	281
The QR Decomposition	283
<b>5 Singular Value Decomposition and the Spectral Theorem</b>	<b>289</b>
<hr/>	
<b>5.1 Singular Value Decomposition of Linear Maps</b>	<b>289</b>
Singular Value Decomposition	289
Uniqueness of Singular Values	293



5.2	Singular Value Decomposition of Matrices	297
	Matrix Version of SVD	297
	SVD and Geometry	301
	Low-rank Approximation	303
5.3	Adjoint Maps	311
	The Adjoint of a Linear Map	311
	Self-adjoint Maps and Matrices	314
	The Four Subspaces	315
	Computing SVD	316
5.4	The Spectral Theorems	320
	Eigenvectors of Self-adjoint Maps and Matrices	321
	Normal Maps and Matrices	324
	Schur Decomposition	327
<b>6</b>	<b>Determinants</b>	<b>333</b>
6.1	Determinants	333
	Multilinear Functions	333
	The Determinant	336
	Existence and Uniqueness of the Determinant	339
6.2	Computing Determinants	346
	Basic Properties	346
	Determinants and Row Operations	349
	Permutations	351
6.3	Characteristic Polynomials	357
	The Characteristic Polynomial of a Matrix	358
	Multiplicities of Eigenvalues	360
	The Cayley–Hamilton Theorem	362
6.4	Applications of Determinants	366
	Volume	366
	Cramer’s Rule	370
	Cofactors and Inverses	371
	<b>Appendix</b>	<b>378</b>
A.1	Sets and Functions	378
	Basic Definitions	378
	Composition and Invertibility	380
A.2	Complex Numbers	382
A.3	Proofs	384
	Logical Connectives	384
	Quantifiers	385
	Contrapositives, Counterexamples, and Proof by Contradiction	386
	Proof by Induction	388

**Contents**

---

xi

<i>Addendum</i>	390
<i>Hints and Answers to Selected Exercises</i>	391
<i>Index</i>	423

# Preface

---

It takes some chutzpah to write a linear algebra book. With so many choices already available, one must ask (and our friends and colleagues did): what is new here?

The most important context for the answer to that question is the intended audience. We wrote the book with our own students in mind; our linear algebra course has a rather mixed audience, including majors in mathematics, applied mathematics, and our joint degree in mathematics and physics, as well as students in computer science, physics, and various fields of engineering. Linear algebra will be fundamental to most if not all of them, but they will meet it in different guises; this course is furthermore the only linear algebra course most of them will take.

Most introductory linear algebra books fall into one of two categories: books written in the style of a freshman calculus text and aimed at teaching students to do computations with matrices and column vectors, or full-fledged “theorem-proof” style rigorous math texts, focusing on abstract vector spaces and linear maps, with little or no matrix computation. This book is different. We offer a unified treatment, building both the basics of computation and the abstract theory from the ground up, emphasizing the connections between the matrix-oriented viewpoint and abstract linear algebraic concepts whenever possible. The result serves students better, whether they are heading into theoretical mathematics or towards applications in science and engineering. Applied math students will learn Gaussian elimination and the matrix form of singular value decomposition (SVD), but they will also learn how abstract inner product space theory can tell them about expanding periodic functions in the Fourier basis. Students in theoretical mathematics will learn foundational results about vector spaces and linear maps, but they will also learn that Gaussian elimination can be a useful and elegant theoretical tool.

Key features of this book include:

- **Early introduction of linear maps:** Our perspective is that mathematicians invented vector spaces so that they could talk about linear maps; for this reason, we introduce linear maps as early as possible, immediately after the introduction of vector spaces.

- **Key concepts referred to early and often:** In general, we have introduced topics we see as central (most notably eigenvalues and eigenvectors) as early as we could, coming back to them again and again as we introduce new concepts which connect to these central ideas. At the end of the course, rather than having just learned the definition of an eigenvector a few weeks ago, students will have worked with the concept extensively throughout the term.
- **Eases the transition from calculus to rigorous mathematics:** Moving beyond the more problem-oriented calculus courses is a challenging transition; the book was written with this transition in mind. It is written in an accessible style, and we have given careful thought to the motivation of new ideas and to parsing difficult definitions and results after stating them formally.
- **Builds mathematical maturity:** Over the course of the book, the style evolves from extremely approachable and example-oriented to something more akin to the style of texts for real analysis and abstract algebra, paving the way for future courses in which a basic comfort with mathematical language and rigor is expected.
- **Fully rigorous, but connects to computation and applications:** This book was written for a proof-based linear algebra course, and contains the necessary theoretical foundation of linear algebra. It also connects that theory to matrix computation and geometry as often as possible; for example, SVD is considered abstractly as the existence of special orthonormal bases for a map; from a geometric point of view emphasizing rotations, reflections, and distortions; and from a more computational point of view, as a matrix factorization. Orthogonal projection in inner product spaces is similarly discussed in theoretical, computational, and geometric ways, and is connected with applied minimization problems such as linear least squares for curve-fitting and approximation of smooth functions on intervals by polynomials.
- **Pedagogical features:** There are various special features aimed at helping students learn to read a mathematics text: frequent “Quick Exercises” serve as checkpoints, with answers upside down at the bottom of the page. Each section ends with a list of “Key Ideas,” summarizing the main points of the section. Features called “Perspectives” at the end of some chapters collect the various viewpoints on important concepts which have been developed throughout the text.
- **Exercises:** The large selection of problems is a mix of the computational and the theoretical, the straightforward and the challenging. There are answers or hints to selected problems in the back of the book.

The book begins with linear systems of equations over  $\mathbb{R}$ , solution by Gaussian elimination, and the introduction of the ideas of pivot variables and free variables. Section 1.3 discusses the geometry of  $\mathbb{R}^n$  and geometric viewpoints on linear systems. We then move into definitions and examples of abstract fields and vector spaces.

Chapter 2 is on linear maps. They are introduced with many examples; the usual cohort of rotations, reflections, projections, and multiplication by matrices in  $\mathbb{R}^n$ , and more abstract examples like differential and integral operators on function spaces. Eigenvalues are first introduced in Section 2.1; the representation of arbitrary linear maps on  $\mathbb{F}^n$  by matrices is proved in Section 2.2. Section 2.3 introduces matrix multiplication as the matrix representation of composition, with an immediate derivation of the usual formula. In Section 2.5, the range, kernel, and eigenspaces of a linear map are introduced. Finally, Section 2.6 introduces the Hamming code as an application of linear algebra over the field of two elements.

Chapter 3 introduces linear dependence and independence, bases, dimension, and the Rank–Nullity Theorem. Section 3.5 introduces coordinates with respect to arbitrary bases and the representation of maps between abstract vector spaces as matrices; Section 3.6 covers change of basis and introduces the idea of diagonalization and its connection to eigenvalues and eigenvectors. Chapter 3 concludes by showing that all matrices over algebraically closed fields can be triangularized.

Chapter 4 introduces general inner product spaces. It covers orthonormal bases and the Gram–Schmidt algorithm, orthogonal projection with applications to least squares and function approximation, normed spaces in general and the operator norm of linear maps and matrices in particular, isometries, and the QR decomposition.

Chapter 5 covers the singular value decomposition and the spectral theorem. We begin by proving the main theorem on the existence of SVD and the uniqueness of singular values for linear maps, then specialize to the matrix factorization. There is a general introduction to adjoint maps and their properties, followed by the Spectral Theorem in the Hermitian and normal cases. Geometric interpretation of SVD and truncations of SVD as low-rank approximation are discussed in Section 5.2. The four fundamental subspaces associated to a linear map, orthogonality, and the connection to the Rank–Nullity Theorem are discussed in Section 5.3.

Finally, Chapter 6 is on determinants. We have taken the viewpoint that the determinant is best characterized as the unique alternating multilinear form on matrices taking value 1 at the identity; we derive many of its properties from that characterization. We introduce the Laplace expansion, give an algorithm for computing determinants via row operations, and prove the sum over permutations formula. The last is presented as a nice example of the power of linear algebra: there is no long digression on combinatorics, but instead permutations are quickly identified with permutation matrices, and concepts like the sign of a permutation arise naturally as familiar linear algebraic constructions. Section 6.3 introduces the characteristic polynomial and the Cayley–Hamilton Theorem, and Section 6.4 concludes the chapter with applications of the determinant to volume and Cramer’s rule.

In terms of student prerequisites, one year of calculus is sufficient. While calculus is not needed for any of the main results, we do rely on it for some examples and exercises (which could nevertheless be omitted). We do not expect students to

have taken a rigorous mathematics course before. The book is written assuming some basic background on sets, functions, and the concept of a proof; there is an appendix containing what is needed for the student's reference (or crash course).

Finally, some thanks are in order. To write a textbook that works in the classroom, it helps to have a classroom to try it out in. We are grateful to the CWRU Math 307 students from Fall 2014, Spring and Fall 2015, and Spring 2016 for their roles as cheerful guinea pigs.

A spectacular feature of the internet age is the ability to get help typesetting a book from someone half-way around the world (where it may in fact be 2 in the morning). We thank the users of [tex.stackexchange.com](http://tex.stackexchange.com) for generously and knowledgeably answering every question we came up with.

We began the project of writing this book while on sabbatical at the Institut de Mathématiques de Toulouse at the University of Toulouse, France. We thank the Institut for its warm hospitality and the Simons Foundation for providing sabbatical support. We also thank the National Science Foundation and the Simons Foundation for additional support.

And lastly, many thanks to Sarah Jarosz, whose album *Build Me Up From Bones* provided the soundtrack for the writing of this book.

ELIZABETH MECKES  
MARK MECKES

*Cleveland, Ohio, USA*

# To the Student

---

This will be one of the most important classes you ever take. Linear algebra and calculus are the foundations of modern mathematics and its applications; the language and viewpoint of linear algebra is so thoroughly woven into the fabric of mathematical reasoning that experienced mathematicians, scientists, and engineers can forget it is there, in the same way that native speakers of a language seldom think consciously about its formal structure. Achieving this fluency is a big part of that nebulous goal of “mathematical maturity.”

In the context of your mathematical education, this book marks an important transition. In it, you will move away from a largely algorithmic, problem-centered viewpoint toward a perspective more consciously grounded in rigorous theoretical mathematics. Making this transition is not easy or immediate, but the rewards of learning to think like a mathematician run deep, no matter what your ultimate career goals are. With that in mind, we wrote this book to be *read* – by you, the student. Reading and learning from an advanced mathematics text book is a skill, and one that we hope this book will help you develop.

There are some specific features of this book aimed at helping you get the most out of it. Throughout the book, you will find “Quick Exercises,” whose answers are usually found (upside down) at the bottom of the page. These are exercises which you should be able to do fairly easily, but for which you may need to write a few lines on the back of an envelope. They are meant to serve as checkpoints; do them! The end of each section lists “Key Ideas,” summarizing (sometimes slightly informally) the big picture of the section. Certain especially important concepts on which there are many important perspectives are summarized in features called “Perspectives” at the end of some chapters. There is an appendix covering the basics of sets, functions, and complex number arithmetic, together with some formal logic and proof techniques. And of course, there are many exercises. Mathematics isn’t something to know, it’s something to do; it is through the exercises that you really learn how.