Linear Algebra

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Linear Algebra

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To Juliette and Peter

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Preface

It takes some chutzpah to write a linear algebra book. With so many choices already available, one must ask (and our friends and colleagues did): what is new here?

The most important context for the answer to that question is the intended audience. We wrote the book with our own students in mind; our linear algebra course has a rather mixed audience, including majors in mathematics, applied mathematics, and our joint degree in mathematics and physics, as well as students in computer science, physics, and various fields of engineering. Linear algebra will be fundamental to most if not all of them, but they will meet it in different guises; this course is furthermore the only linear algebra course most of them will take.

Most introductory linear algebra books fall into one of two categories: books written in the style of a freshman calculus text and aimed at teaching students to do computations with matrices and column vectors, or full-fledged "theorem–proof" style rigorous math texts, focusing on abstract vector spaces and linear maps, with little or no matrix computation. This book is different. We offer a unified treatment, building both the basics of computation and the abstract theory from the ground up, emphasizing the connections between the matrix-oriented viewpoint and abstract linear algebraic concepts whenever possible. The result serves students better, whether they are heading into theoretical mathematics or towards applications in science and engineering. Applied math students will learn Gaussian elimination and the matrix form of singular value decomposition (SVD), but they will also learn how abstract inner product space theory can tell them about expanding periodic functions in the Fourier basis. Students in theoretical mathematics will learn foundational results about vector spaces and linear maps, but they will also learn that Gaussian elimination can be a useful and elegant theoretical tool.

Key features of this book include:

- **Early introduction of linear maps:** Our perspective is that mathematicians invented vector spaces so that they could talk about linear maps; for this reason, we introduce linear maps as early as possible, immediately after the introduction of vector spaces.

- **Key concepts referred to early and often:** In general, we have introduced topics we see as central (most notably eigenvalues and eigenvectors) as early as we could, coming back to them again and again as we introduce new concepts which connect to these central ideas. At the end of the course, rather than having just learned the definition of an eigenvector a few weeks ago, students will have worked with the concept extensively throughout the term.
- **Eases the transition from calculus to rigorous mathematics:** Moving beyond the more problem-oriented calculus courses is a challenging transition; the book was written with this transition in mind. It is written in an accessible style, and we have given careful thought to the motivation of new ideas and to parsing difficult definitions and results after stating them formally.
- **Builds mathematical maturity:** Over the course of the book, the style evolves from extremely approachable and example-oriented to something more akin to the style of texts for real analysis and abstract algebra, paving the way for future courses in which a basic comfort with mathematical language and rigor is expected.
- **Fully rigorous, but connects to computation and applications:** This book was written for a proof-based linear algebra course, and contains the necessary theoretical foundation of linear algebra. It also connects that theory to matrix computation and geometry as often as possible; for example, SVD is considered abstractly as the existence of special orthonormal bases for a map; from a geometric point of view emphasizing rotations, reflections, and distortions; and from a more computational point of view, as a matrix factorization. Orthogonal projection in inner product spaces is similarly discussed in theoretical, computational, and geometric ways, and is connected with applied minimization problems such as linear least squares for curve-fitting and approximation of smooth functions on intervals by polynomials.
- **Pedagogical features:** There are various special features aimed at helping students learn to read a mathematics text: frequent "Quick Exercises" serve as checkpoints, with answers upside down at the bottom of the page. Each section ends with a list of "Key Ideas," summarizing the main points of the section. Features called "Perspectives" at the end of some chapters collect the various viewpoints on important concepts which have been developed throughout the text.
- **Exercises:** The large selection of problems is a mix of the computational and the theoretical, the straightforward and the challenging. There are answers or hints to selected problems in the back of the book.

The book begins with linear systems of equations over $\mathbb{R}$, solution by Gaussian elimination, and the introduction of the ideas of pivot variables and free variables. Section 1.3 discusses the geometry of $\mathbb{R}^n$ and geometric viewpoints on linear systems. We then move into definitions and examples of abstract fields and vector spaces.

Chapter 2 is on linear maps. They are introduced with many examples; the usual cohort of rotations, reflections, projections, and multiplication by matrices in $\mathbb{R}^n$, and more abstract examples like differential and integral operators on function spaces. Eigenvalues are first introduced in Section 2.1; the representation of arbitrary linear maps on $\mathbb{F}^n$ by matrices is proved in Section 2.2. Section 2.3 introduces matrix multiplication as the matrix representation of composition, with an immediate derivation of the usual formula. In Section 2.5, the range, kernel, and eigenspaces of a linear map are introduced. Finally, Section 2.6 introduces the Hamming code as an application of linear algebra over the field of two elements.

Chapter 3 introduces linear dependence and independence, bases, dimension, and the Rank–Nullity Theorem. Section 3.5 introduces coordinates with respect to arbitrary bases and the representation of maps between abstract vector spaces as matrices; Section 3.6 covers change of basis and introduces the idea of diagonalization and its connection to eigenvalues and eigenvectors. Chapter 3 concludes by showing that all matrices over algebraically closed fields can be triangularized.

Chapter 4 introduces general inner product spaces. It covers orthonormal bases and the Gram–Schmidt algorithm, orthogonal projection with applications to least squares and function approximation, normed spaces in general and the operator norm of linear maps and matrices in particular, isometries, and the QR decomposition.

Chapter 5 covers the singular value decomposition and the spectral theorem. We begin by proving the main theorem on the existence of SVD and the uniqueness of singular values for linear maps, then specialize to the matrix factorization. There is a general introduction to adjoint maps and their properties, followed by the Spectral Theorem in the Hermitian and normal cases. Geometric interpretation of SVD and truncations of SVD as low-rank approximation are discussed in Section 5.2. The four fundamental subspaces associated to a linear map, orthogonality, and the connection to the Rank–Nullity Theorem are discussed in Section 5.3.

Finally, Chapter 6 is on determinants. We have taken the viewpoint that the determinant is best characterized as the unique alternating multilinear form on matrices taking value 1 at the identity; we derive many of its properties from that characterization. We introduce the Laplace expansion, give an algorithm for computing determinants via row operations, and prove the sum over permutations formula. The last is presented as a nice example of the power of linear algebra: there is no long digression on combinatorics, but instead permutations are quickly identified with permutation matrices, and concepts like the sign of a permutation arise naturally as familiar linear algebraic constructions. Section 6.3 introduces the characteristic polynomial and the Cayley–Hamilton Theorem, and Section 6.4 concludes the chapter with applications of the determinant to volume and Cramer's rule.

In terms of student prerequisites, one year of calculus is sufficient. While calculus is not needed for any of the main results, we do rely on it for some examples and exercises (which could nevertheless be omitted). We do not expect students to have taken a rigorous mathematics course before. The book is written assuming some basic background on sets, functions, and the concept of a proof; there is an appendix containing what is needed for the student's reference (or crash course).

Finally, some thanks are in order. To write a textbook that works in the classroom, it helps to have a classroom to try it out in. We are grateful to the CWRU Math 307 students from Fall 2014, Spring and Fall 2015, and Spring 2016 for their roles as cheerful guinea pigs.

A spectacular feature of the internet age is the ability to get help typesetting a book from someone half-way around the world (where it may in fact be 2 in the morning). We thank the users of tex.stackexchange.com for generously and knowledgeably answering every question we came up with.

We began the project of writing this book while on sabbatical at the Institut de Mathématiques de Toulouse at the University of Toulouse, France. We thank the Institut for its warm hospitality and the Simons Foundation for providing sabbatical support. We also thank the National Science Foundation and the Simons Foundation for additional support.

And lastly, many thanks to Sarah Jarosz, whose album *Build Me Up From Bones* provided the soundtrack for the writing of this book.

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To the Student

This will be one of the most important classes you ever take. Linear algebra and calculus are the foundations of modern mathematics and its applications; the language and viewpoint of linear algebra is so thoroughly woven into the fabric of mathematical reasoning that experienced mathematicians, scientists, and engineers can forget it is there, in the same way that native speakers of a language seldom think consciously about its formal structure. Achieving this fluency is a big part of that nebulous goal of "mathematical maturity."

In the context of your mathematical education, this book marks an important transition. In it, you will move away from a largely algorithmic, problem-centered viewpoint toward a perspective more consciously grounded in rigorous theoretical mathematics. Making this transition is not easy or immediate, but the rewards of learning to think like a mathematician run deep, no matter what your ultimate career goals are. With that in mind, we wrote this book to be *read* – by you, the student. Reading and learning from an advanced mathematics text book is a skill, and one that we hope this book will help you develop.

There are some specific features of this book aimed at helping you get the most out of it. Throughout the book, you will find "Quick Exercises," whose answers are usually found (upside down) at the bottom of the page. These are exercises which you should be able to do fairly easily, but for which you may need to write a few lines on the back of an envelope. They are meant to serve as checkpoints; do them! The end of each section lists "Key Ideas," summarizing (sometimes slightly informally) the big picture of the section. Certain especially important concepts on which there are many important perspectives are summarized in features called "Perspectives" at the end of some chapters. There is an appendix covering the basics of sets, functions, and complex number arithmetic, together with some formal logic and proof techniques. And of course, there are many exercises. Mathematics isn't something to know, it's something to do; it is through the exercises that you really learn how.