# 1

# Linear Systems and Vector Spaces

## 1.1 Linear Systems of Equations

### Bread, Beer, and Barley

We begin with a very simple example. Suppose you have 20 pounds of raw barley and you plan to turn some of it into bread and some of it into beer. It takes one pound of barley to make a loaf of bread and a quarter pound of barley to make a pint of beer. You could use up all the barley on 20 loaves of bread, or alternatively, on 80 pints of beer (although that's probably not advisable). What are your other options? Before rolling up our sleeves and figuring that out, we make the following obvious-seeming observation, without which rolling up our sleeves won't help much.

> It's very difficult to talk about something that has no name.

That is, before we can *do math* we have to have something concrete to *do math to*. Therefore: let $x$ be the number of loaves of bread you plan to bake and $y$ be the number of pints of beer you want to wash it down with. Then the information above can be expressed as

$$x + \frac{1}{4}y = 20, \tag{1.1}$$

and now we have something real (to a mathematician, anyway) to work with.

This object is called a **linear equation in two variables**. Here are some things to notice about it:

- There are infinitely many solutions to equation (1.1) (assuming you're okay with fractional loaves of bread and fractional pints of beer).
- We only care about the positive solutions, but even so, there are still infinitely many choices. (It's important to notice that our interest in positive solutions is a feature of the real-world situation being modeled, but it's not built into the model itself. We just have to remember what we're doing when interpreting solutions. This caveat may or may not be true of other models.)

- We could specify how much bread we want and solve for how much beer we can have, or vice versa. Or we could specify a fixed ratio of bread to beer (i.e., fix the value of $c = x/y$) and solve.
- For the graphically inclined, we can draw a picture of all the solutions of this equation in the $x$–$y$ plane, as follows:

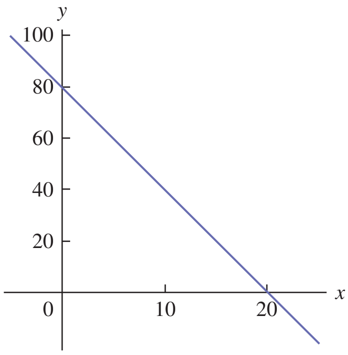


**Figure 1.1** Graph of the solutions of equation (1.1).

Each point in the $x$–$y$ plane corresponds to a quantity of bread and of beer, and if a point lies on the line above, it means that it is possible to exactly use up all of the barley by making those quantities of bread and beer.

We have, of course, drastically oversimplified the situation. For starters, you also need yeast to make both bread and beer. It takes 2 teaspoons yeast to make a loaf of bread and a quarter teaspoon to make a pint of beer. Suppose you have meticulously measured that you have exactly 36 teaspoons of yeast available for your fermentation processes. We now have what's called a **linear system of equations**, as follows:

$$\begin{aligned} x + \frac{1}{4}y &= 20 \\ 2x + \frac{1}{4}y &= 36. \end{aligned} \tag{1.2}$$

You could probably come up with a couple of ways to solve this system. Just to give one, if we subtract the first equation from the second, we get

$$x = 16.$$

If we then plug $x = 16$ into the first equation and solve for $y$, we get

$$16 + \frac{1}{4}y = 20 \qquad \Longleftrightarrow \qquad y = 16.$$

(The symbol $\Longleftrightarrow$ above is read aloud as "if and only if," and it means that the two equations are **equivalent**; i.e., that any given value of $y$ makes the equation on the left true *if and only if* it makes the equation on the right true. Please learn to use this symbol when appropriate; it is not correct to use an equals sign instead.)

Cambridge University Press & Assessment
978-1-107-17790-1 — Linear Algebra
Elizabeth S. Meckes , Mark W. Meckes
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We now say that the **solution** to the linear system (1.2) is $x = 16, y = 16$. In particular, we've discovered that there's exactly one way to use up all the barley and all the yeast to make bread and beer.

Here are some things to notice:

- We can represent the situation modeled by system (1.2) graphically:

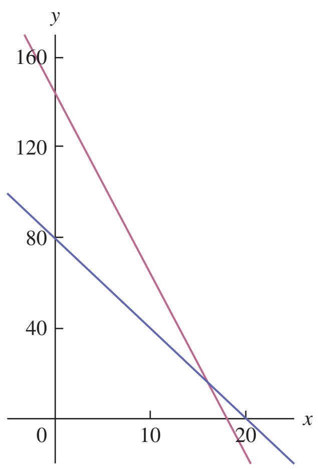


**Figure 1.2** Graph of the solutions of the equations in system (1.2): blue for the first equation and red for the second.

  In Figure 1.2, each line represents all the solutions to one of the equations in system (1.2). The intersection of these two lines (at the point (16, 16)) represents the unique way of using up all of both your ingredients.
- If the amount of yeast had been different, we might have ended up with a solution that did not have both $x, y > 0$.

> **Quick Exercise #1.** Give a quantity of yeast for which the corresponding system has a solution, but with either $x < 0$ or $y < 0$.

- If you switch to a different bread recipe that only requires 1 teaspoon (tsp) of yeast per loaf, we might instead have infinitely many solutions (as we did before considering the yeast), or none. (How?)

Suppose now that your menu gets more interesting: someone comes along with milk and dried rosemary. Your bread would taste better with a little of each put into the dough; also, you can use the milk to make a simple cheese, which would also be nice if you flavored it with rosemary. The beer might be good flavored with rosemary, too.* Suppose you use 1 cup of milk per loaf of bread, and 8 cups

*What great philosopher of the modern era said "A man can live on packaged food from here 'til Judgment Day if he's got enough rosemary."?

**QA #1:** Anything less than 20 teaspoons of yeast will result in $x < 0$.

of milk per round of cheese. You put 2 tsps of rosemary in each loaf of bread, 1 tsp in each round of cheese and 1/4 tsp in each pint of beer. Then we have a new variable $z$ – the number of rounds of cheese – and two new equations, one for the milk and one for the rosemary. Suppose you have 11 gallons (i.e., 176 cups) of milk and 56 tsps of rosemary. Our linear system now becomes:

$$\begin{aligned} x + \frac{1}{4}y &= 20 \\ 2x + \frac{1}{4}y &= 36 \\ x + 8z &= 176 \\ 2x + \frac{1}{4}y + z &= 56. \end{aligned} \tag{1.3}$$

If we go ahead and solve the system, we will find that $x = 16, y = 16, z = 20$. Since we've been given the solution, though, it's quicker just to check that it's correct.

**Quick Exercise #2.** Check that this solution is correct.

In any case, it seems rather lucky that we could solve the system at all; i.e., with four ingredients being divided among three products, we were able to exactly use up everything. A moment's reflection confirms that this was a lucky accident: since it worked out so perfectly, we can see that if you'd had more rosemary but the same amount of everything else, you wouldn't be able to use up everything exactly. This is a first example of a phenomenon we will meet often: *redundancy*. The system of equations (1.3) is redundant: if we call the equations $E_1, E_2, E_3, E_4$, then

$$E_4 = \left(\frac{1}{8}\right) E_1 + \left(\frac{7}{8}\right) E_2 + \left(\frac{1}{8}\right) E_3.$$

In particular, if $(x, y, z)$ is a solution to all three equations $E_1$, $E_2$, and $E_3$, then it satisfies $E_4$ automatically. Thus $E_4$ tells us nothing about the values of $(x, y, z)$ that we couldn't already tell from just the first three equations. The solution $x = 16, y = 16, z = 20$ is in fact the unique solution to the first three equations, and it satisfies the redundant fourth equation for free.

## Linear Systems and Solutions

Now we'll start looking at more general systems of equations that resemble the ones that came up above. Later in this book, we will think about various types of numbers, some of which you may never have met. But for now, we will restrict

**QA #2:** If you did anything more complicated than just plug in the numbers 16, 16, and 20, then you're making it too hard.

our attention to equations involving real numbers and real variables, i.e., unknown real numbers.

**Definition** The set of all real numbers is denoted by $\mathbb{R}$. The notation $t \in \mathbb{R}$ is read "$t$ is in $\mathbb{R}$" or "$t$ is an element of $\mathbb{R}$" or simply "$t$ is a real number."

**Definition** A **linear system of $m$ equations in $n$ variables over** $\mathbb{R}$ is a set of equations of the form

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m. \end{aligned} \tag{1.4}$$

Here, $a_{ij} \in \mathbb{R}$ for each pair $(i, j)$ with $1 \leq i \leq m$ and $1 \leq j \leq n$; $b_i \in \mathbb{R}$ for each $1 \leq i \leq m$; and $x_1, \ldots, x_n$ are the $n$ variables of the system. Such a system of equations is also sometimes called an $m \times n$ **linear system.**

A **solution** of the linear system (1.4) is a set of real numbers $c_1, \ldots, c_n$ such that

$$\begin{aligned} a_{11}c_1 + \cdots + a_{1n}c_n &= b_1 \\ &\vdots \\ a_{m1}c_1 + \cdots + a_{mn}c_n &= b_m. \end{aligned}$$

That is, a solution is a list of values that you can plug in for $x_1, \ldots, x_n$ so that the equations in system (1.4) are satisfied.

What makes a system *linear* is that only a couple simple things can be done to the variables: they are multiplied by constants, and added to other variables multiplied by constants. The variables aren't raised to powers, multiplied together, or plugged into complicated functions like logarithms or cosines.

A crucial thing to notice about the definition above is that a solution is defined to be "something that works," and *not* "something you find by 'solving.'" That is, if it acts like a solution, it's a solution – it doesn't matter where you got it. For example, if you are asked to verify that $x = 1, y = -1$ is a solution to the linear system

$$\begin{aligned} 3x + 2y &= 1 \\ x - y &= 2, \end{aligned}$$

all you need to do is plug in $x = 1, y = -1$ and see that the equations are both true. You *do not* need to solve the system and then observe that what you found is that $x = 1, y = -1$. (This point was mentioned earlier in the answer to Quick

Exercise #2.) Doing that in this case won't have resulted in any great harm, just wasted a few minutes of your time. We could have cooked up ways to waste a lot more of your time, though.

This is an instance of the following central principle of mathematics.

**The Rat Poison Principle**

Q: How do you find out if something is rat poison?

A: You feed it to a rat.

The point is that if someone hands you something and asks if it is rat poison, the answer doesn't depend on where or how she got it. The crucial thing is to see whether it does the job. The Rat Poison Principle is important because it tells you what to do in abstract situations when solving isn't an option.

**Example** Suppose that we know that $x_1 = c_1, \ldots, x_n = c_n$ is a solution to the linear system

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= 0. \end{aligned} \tag{1.5}$$

(A system like this, in which all the constants on the right-hand side are 0, is called **homogeneous**.) This simply means that

$$\begin{aligned} a_{11}c_1 + \cdots + a_{1n}c_n &= 0 \\ &\vdots \\ a_{m1}c_1 + \cdots + a_{mn}c_n &= 0. \end{aligned}$$

Now if $r \in \mathbb{R}$ is any constant, then

$$\begin{aligned} a_{11}(rc_1) + \cdots + a_{1n}(rc_n) &= r(a_{11}c_1 + \cdots + a_{1n}c_n) = r0 = 0 \\ &\vdots \\ a_{m1}(rc_1) + \cdots + a_{mn}(rc_n) &= r(a_{m1}c_1 + \cdots + a_{mn}c_n) = r0 = 0. \end{aligned}$$

This tells us that $x_1 = rc_1, \ldots, x_n = rc_n$ is another solution of the system (1.5). So, without solving anything, we've found a way to take one known solution of this linear system and come up with infinitely many different solutions. ▲

We end this section with some more terminology that will be important in talking about linear systems and their solutions.

**Definition** A linear system is called **consistent** if a solution exists and is called **inconsistent** if it has no solution.

A solution $c_1, \dots, c_n$ of an $m \times n$ linear system is called **unique** if it is the *only* solution; i.e., if whenever $c'_1, \dots, c'_n$ is a solution to the same system, $c_i = c'_i$ for $1 \leq i \leq n$.

**Quick Exercise #3.** Which of the three systems (1.1) (a $1 \times 2$ system), (1.2), (1.3) are consistent? Which have unique solutions?

### KEY IDEAS

- A linear system of equations over $\mathbb{R}$ is a set of equations of the form

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

  The list of numbers $c_1, \dots, c_n$ is a solution to the system if you can plug each $c_i$ in for $x_i$ in the equations above and the resulting equations are all true.
- The Rat Poison Principle: if you feed it to a rat and the rat dies, it's rat poison. Things in math are often defined by what they do.
- A linear system is consistent if it has at least one solution; otherwise it is inconsistent.
- A solution to a linear system is unique if it is the only solution.

## EXERCISES

1.1.1 Suppose that you're making bread, beer, and cheese using the recipes in this section. As before, you have 176 cups of milk and 56 tsps of rosemary, but you now have only 35 tsps of yeast and as much barley as you need.

(a) How much bread, beer, and cheese can you make to exactly use up your yeast, milk, and rosemary?

(b) How much barley will you need?

1.1.2 For each of the following linear systems, graph the set of solutions of each equation. Is the system consistent or inconsistent? If it is consistent, does it have a unique solution?

**QA #3:** All three are consistent, as we've seen. We've also seen that the system (1.2) has a unique solution but the equation (1.1) does not. We simply told you that the solution of (1.3) is unique; we'll see why that's true in the next section.

(a) $\begin{aligned} 2x - y &= 7 \\ x + y &= 2 \end{aligned}$ (b) $\begin{aligned} 3x + 2y &= 1 \\ x - y &= -3 \\ 2x + y &= 0 \end{aligned}$ (c) $\begin{aligned} x + y &= 2 \\ 2x - y &= 1 \\ x - y &= -1 \end{aligned}$

(d) $\begin{aligned} 4x - 2y &= -6 \\ -2x + y &= 2 \end{aligned}$ (e) $\begin{aligned} x + 2y - z &= 0 \\ -x + y + z &= 0 \\ x - y - z &= 0 \end{aligned}$ (f) $\begin{aligned} x + y + z &= 0 \\ x - y + z &= 0 \\ x + y - z &= 0 \end{aligned}$

**1.1.3** For each of the following linear systems, graph the set of solutions of each equation. Is the system consistent or inconsistent? If it is consistent, does it have a unique solution?

(a) $\begin{aligned} x + 2y &= 5 \\ 2x + y &= 0 \end{aligned}$ (b) $\begin{aligned} x - y &= -1 \\ 2x - y &= 4 \\ x + y &= 3 \end{aligned}$ (c) $\begin{aligned} -2x + y &= 0 \\ x + y &= -3 \\ x - y &= 1 \end{aligned}$

(d) $\begin{aligned} 2x - 4y &= 2 \\ -x + 2y &= -1 \end{aligned}$ (e) $\begin{aligned} 0x + 0y + z &= 0 \\ 0x + y + 0z &= 0 \\ x + 0y + 0z &= 0 \end{aligned}$ (f) $\begin{aligned} 0x + 0y + z &= 0 \\ x + 0y + z &= 1 \\ x + 0y - z &= -1 \end{aligned}$

**1.1.4** Modify the redundant linear system (1.3) to make it inconsistent. Interpret your new system in terms of the effort to use up all of the ingredients when making the bread, beer, and cheese.

**1.1.5** Give geometric descriptions of the sets of all solutions for each of the following linear systems.

(a) $0x + 0y + z = 0$ (b) $\begin{aligned} 0x + 0y + z &= 0 \\ 0x + y + 0z &= 0 \end{aligned}$ (c) $\begin{aligned} 0x + 0y + z &= 0 \\ 0x + y + 0z &= 0 \\ x + 0y + 0z &= 0 \end{aligned}$

(d) $\begin{aligned} 0x + 0y + z &= 0 \\ 0x + y + 0z &= 0 \\ x + 0y + 0z &= 0 \\ x + y + z &= 0 \end{aligned}$ (e) $\begin{aligned} 0x + 0y + z &= 0 \\ 0x + y + 0z &= 0 \\ x + 0y + 0z &= 0 \\ x + y + z &= 1 \end{aligned}$

**1.1.6** Give geometric descriptions of the sets of all solutions for each of the following linear systems.

(a) $0x + y + 0z = 1$ (b) $\begin{aligned} x + 0y + 0z &= 0 \\ 0x + y + 0z &= 0 \end{aligned}$ (c) $\begin{aligned} 0x + y + z &= 0 \\ x + 0y + z &= 0 \\ x + y + 0z &= 0 \end{aligned}$

(d) $\begin{aligned} 0x + y + z &= 0 \\ x + 0y + z &= 0 \\ x + y + 0z &= 0 \\ x + y + z &= 0 \end{aligned}$ (e) $\begin{aligned} 0x + y + z &= 0 \\ x + 0y + z &= 0 \\ x + y + 0z &= 0 \\ x + y + z &= 1 \end{aligned}$

**1.1.7** Suppose that $f(x) = ax^2 + bx + c$ is a quadratic polynomial whose graph passes through the points $(-1, 1)$, $(0, 0)$, and $(1, 2)$.
(a) Find a linear system satisfied by $a$, $b$, and $c$.
(b) Solve the linear system to determine what the function $f$ is.

**1.1.8** Let $f(x) = a + b\cos x + c\sin x$ for some $a, b, c \in \mathbb{R}$. (This is a simple example of a **trigonometric polynomial**.) Suppose you know that

$$f(0) = -1, \qquad f(\pi/2) = 2, \qquad \text{and} \qquad f(\pi) = 3.$$

(a) Find a linear system satisfied by $a$, $b$, and $c$.
(b) Solve the linear system to determine what the function $f$ is.

**1.1.9** Show that $\left(\dfrac{de - bf}{ad - bc}, \dfrac{af - ec}{ad - bc}\right)$ is a solution of the $2 \times 2$ linear system

$$\begin{aligned} ax + by &= e \\ cx + dy &= f, \end{aligned}$$

as long as $ad \neq bc$.

**1.1.10** Suppose that $x_1 = c_1, \ldots, x_n = c_n$ and $x_1 = d_1, \ldots, x_n = d_n$ are both solutions of the linear system (1.4). Under what circumstances is $x_1 = c_1 + d_1, \ldots, x_n = c_n + d_n$ also a solution?

**1.1.11** Suppose that $x_1 = c_1, \ldots, x_n = c_n$ and $x_1 = d_1, \ldots, x_n = d_n$ are both solutions of the linear system (1.4), and $t$ is a real number. Show that $x_1 = tc_1 + (1 - t)d_1, \ldots, x_n = tc_n + (1 - t)d_n$ is also a solution to system (1.4).

## 1.2 Gaussian Elimination

### The Augmented Matrix of a Linear System

In the last section we solved some small linear systems in *ad hoc* ways; i.e., we weren't particularly systematic about it. You can easily imagine that if you're going to try to solve a much larger system by hand, or (more realistically) if you're going to program a computer to do the algebra for you, you need a more systematic approach.

The first thing we'll need is a standardized way to write down a linear system. For a start, we should treat all the variables the same way in every equation. That means writing them in the same order, and including all of the variables in every equation. If one of the equations as originally written doesn't contain one of the variables, we can make it appear by including it with a coefficient of 0. In the examples in Section 1.1 we already kept the variables in a consistent order, but didn't always have all the variables written in every equation. So, for example, we should rewrite the system (1.3) as

$$\begin{aligned} x + \frac{1}{4}y + 0z &= 20 \\ 2x + \frac{1}{4}y + 0z &= 36 \\ x + 0y + 8z &= 176 \\ 2x + \frac{1}{4}y + z &= 56. \end{aligned} \tag{1.6}$$

The next thing to notice is that there's a lot of notation which we don't actually need, as long as we remember that what we're working with is a linear system. The coefficients on the left, and the constants on the right, contain all the information about the system, and we can completely do away with the variable names and the $+$ and $=$ signs. We *do*, however, need to keep track of the relative positions of those numbers. We do this by writing them down in what's called a matrix.

**Definition** Let $m$ and $n$ be natural numbers. An $m \times n$ **matrix** A over $\mathbb{R}$ is a doubly indexed collection of real numbers

$$\mathrm{A} = \left[a_{ij}\right]_{\substack{1 \le i \le m \\ 1 \le j \le n}}.$$

That is, for each ordered pair of integers $(i, j)$ with $i \in \{1, 2, \ldots, m\}$ and $j \in \{1, 2, \ldots, n\}$, there is a corresponding real number $a_{ij}$. The number $a_{ij}$ is called the $(i, j)$ **entry** of A. We sometimes denote the $(i, j)$ entry of A by $\left[\mathrm{A}\right]_{ij}$.

We denote the set of $m \times n$ matrices over $\mathbb{R}$ by $\mathrm{M}_{m,n}(\mathbb{R})$.

One way a (small) matrix can be specified is by just writing down all of its entries, like this:

$$\mathrm{A} = \begin{bmatrix} 1 & \frac{1}{4} & 0 \\ 2 & \frac{1}{4} & 0 \\ 1 & 0 & 8 \\ 2 & \frac{1}{4} & 1 \end{bmatrix}. \tag{1.7}$$

When we do this, an $m \times n$ matrix has $m$ rows and $n$ columns, and the $(i, j)$ entry $a_{ij}$ appears in the $i$th row (from the top) and the $j$th column (from the left). For example, the matrix above is a $4 \times 3$ matrix, and $a_{21} = 2$. To remember which is $m$ and which is $n$, and which is $i$ and which is $j$, keep in mind:

Rows, then columns.

The matrix in equation (1.7) is called the **coefficient matrix** of the linear system (1.6). In order to record all of the information in a linear system, we need to