CAMBRIDGE TRACTS IN MATHEMATICS

208 Representations of Elementary Abelian *p*-Groups and Vector Bundles

CAMBRIDGE TRACTS IN MATHEMATICS

A complete list of books in the series can be found at www.cambridge.org/mathematics. Recent titles include the following:

173. Enumeration of Finite Groups. By S. R. BLACKBURN, P. M. NEUMANN, and G. VENKATARAMAN
174. Forcing Idealized. By J. ZAPLETAL
175. The Large Sieve and its Applications. By E. KOWALSKI
176. The Monster Group and Majorana Involutions. By A. A. IVANOV
177. A Higher-Dimensional Sieve Method. By H. G. DIAMOND, H. HALBERSTAM, and W. F. GALWAY
178. Analysis in Positive Characteristic. By A. N. KOCHUBEI
179. Dynamics of Linear Operators. By F. BAYART and É. MATHERON
180. Synthetic Geometry of Manifolds. By A. KOCK
181. Totally Positive Matrices. By A. PINKUS
182. Nonlinear Markov Processes and Kinetic Equations. By V. N. KOLOKOLTSOV
183. Period Domains over Finite and p-adic Fields. By J.-F. DAT, S. ORLIK, and M. RAPOPORT
184. Algebraic Theories. By J. ADÁMEK, J. ROSICKÝ, and E. M. VITALE
185. Rigidity in Higher Rank Abelian Group Actions I: Introduction and Cocycle Problem. By A. KATOK and V. NIŢICĂ
186. Dimensions, Embeddings, and Attractors. By J. C. ROBINSON
187. Convexity: An Analytic Viewpoint. By B. SIMON
188. Modern Approaches to the Invariant Subspace Problem. By I. CHALENDAR and J. R. PARTINGTON
189. Nonlinear Perron–Frobenius Theory. By B. LEMMENS and R. NUSSBAUM
190. Jordan Structures in Geometry and Analysis. By C.-H. CHU
191. Malliavin Calculus for Lévy Processes and Infinite-Dimensional Brownian Motion. By H. OSSWALD
192. Normal Approximations with Malliavin Calculus. By I. NOURDIN and G. PECCATI
193. Distribution Modulo One and Diophantine Approximation. By Y. BUGEAUD
194. Mathematics of Two-Dimensional Turbulence. By S. KUKSIN and A. SHIRIKYAN
195. A Universal Construction for Groups Acting Freely on Real Trees. By I. CHISWELL and T. MÜLLER
196. The Theory of Hardy's Z-Function. By A. IVIĆ
197. Induced Representations of Locally Compact Groups. By E. KANIUTH and K. F. TAYLOR
198. Topics in Critical Point Theory. By K. PERERA and M. SCHECHTER
199. Combinatorics of Minuscule Representations. By R. M. GREEN
200. Singularities of the Minimal Model Program. By J. KOLLÁR
201. Coherence in Three-Dimensional Category Theory. By N. GURSKI
202. Canonical Ramsey Theory on Polish Spaces. By V. KANOVEI, M. SABOK, and J. ZAPLETAL
203. A Primer on the Dirichlet Space. By O. EL-FALLAH, K. KELLAY, J. MASHREGHI, and T. RANSFORD
204. Group Cohomology and Algebraic Cycles. By B. TOTARO
205. Ridge Functions. By A. PINKUS
206. Probability on Real Lie Algebras. By U. FRANZ and N. PRIVAULT
207. Auxiliary Polynomials in Number Theory. By D. MASSER
208. Representations of Elementary Abelian p-Groups and Vector Bundles. By D. J. BENSON

Cambridge University Press
978-1-107-17417-7 — Representations of Elementary Abelian p-Groups and Vector Bundles
David J. Benson
Frontmatter
More Information

Representations of Elementary Abelian *p*-Groups and Vector Bundles

DAVID J. BENSON
University of Aberdeen

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

4843/24, 2nd Floor, Ansari Road, Daryaganj, Delhi – 110002, India

79 Anson Road, #06–04/06, Singapore 079906

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning, and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9781107174177
DOI: 10.1017/9781316795699

First published 2017

A catalogue record for this publication is available from the British Library.

Library of Congress Cataloging-in-Publication Data
Names: Benson, D. J. (David J.), 1955–
Title: Representations of elementary abelian p-groups and vector bundles / David J. Benson, University of Aberdeen.
Description: Cambridge : Cambridge University Press, [2017] |
Series: Cambridge tracts in mathematics ; 208 |
Includes bibliographical references and index.
Identifiers: LCCN 2016046127 | ISBN 9781107174177 (alk. paper)
Subjects: LCSH: Abelian p-groups. | Abelian groups. | Vector bundles.
Classification: LCC QA180 .B46 2017 | DDC 512/.25–dc23
LC record available at https://lccn.loc.gov/2016046127

ISBN 978-1-107-17417-7 Hardback

Contents

Preface

The origins of this book lie in an extended visit that I made in the Spring of 2008 to MSRI in Berkeley, California as the Simons Professor for the programme on Representation Theory of Finite Groups and Related Topics. Jon Carlson and Julia Pevtsova were there for a large part of this time, and Eric Friedlander passed through town several times. Through numerous conversations and lectures, they ignited my interest in the theory of modules of constant Jordan type. At first, I was reluctant to be drawn in. But then I managed to prove one of the conjectures from a paper of Carlson, Friedlander and Pevtsova [81], and I was hooked. This work has been published [42] and appears here as Section 5.12.

After that initial success, I started working with Julia Pevtsova on the vector bundles on projective space associated to modules of constant Jordan type. We refined a previous definition of Friedlander and Pevtsova, and proved a realisation theorem. This work has also been published [53] and appears here as Theorem 8.1.1, Sections 8.2–8.9 and Section 10.2.

I wrote several other papers soon after that, and soon it was getting to the point where I had so much material that it made sense to make a book based on the resulting series of papers as well as a great deal of other unpublished work. This is that book.

My thanks go to MSRI for its hospitality in 2008 and 2013 while parts of this work were being written; David Eisenbud for sharing his knowledge of vector bundles on projective spaces; Jon Carlson, Eric Friedlander and Julia Pevtsova for numerous conversations; Serge Bouc, Radu Stancu and Jon Carlson for their extensive feedback on earlier versions of the text; Jeremy Rickard for various interesting comments and questions, and particularly for formulating Conjecture 5.13.1; Andrew Granville for helping me with some number theoretic questions, and especially for formulating and supplying a proof of a statement similar to Lemma 12.10.1. I'd also like to thank Jesse Burke, Jon Carlson, Claudia Miller, Julia Pevtsova and Greg Stevenson for enlightening conversations and feedback related to Chapter 11, and Mark Walker for giving a talk at a conference in Seattle that stimulated my interest in Orlov's correspondence.

Cambridge University Press
978-1-107-17417-7 — Representations of Elementary Abelian p-Groups and Vector Bundles
David J. Benson
Frontmatter
More Information

Introduction

Let us begin with an example. Consider the following 30×30 matrices with entries in an algebraically closed field k of characteristic two.

$$
\left(\begin{array}{cccccccccccccccccccccccccccccc}
&&&&& &&&&& &&&&& &&&&& &&&&& &&&& \\
&&&&& &&&&& &&&&& &&&&& &&&&& &&&& \\
a &&&&& &&&&& &&&&& &&&&& &&&&& &&&& \\
b &&&&& &&&&& &&&&& &&&&& &&&&& &&&& \\
c &&&&& &&&&& &&&&& &&&&& &&&&& &&&& \\
d &&&&& &&&&& &&&&& &&&&& &&&&& &&&& \\
e &&&&& &&&&& &&&&& &&&&& &&&&& &&&& \\
f &&&&& &&&&& &&&&& &&&&& &&&&& &&&& \\
& a &&&&& f & e &&&&& &&&&& &&&&& &&&&& && \\
& b &&&& f && d &&&&& &&&&& &&&&& &&&&& && \\
& c &&&& e & d &&&&& &&&&& &&&&& &&&&& &&& \\
& d && c & b &&&&& &&&&& &&&&& &&&&& &&&&& \\
& e & c && a &&&&& &&&&& &&&&& &&&&& &&&&& \\
& f & b & a &&&&& &&&&& &&&&& &&&&& &&&&& & \\
&& d && f & a && c &&&&& &&&&& &&&&& &&&&& && \\
&& e &&&& a &&&&& &&&&& &&&&& &&&&& &&& \\
&& f &&&&& a &&&&& &&&&& &&&&& &&&&& && \\
&&& d && b &&&&& &&&&& &&&&& &&&&& &&&& \\
&&& e & f && b & c &&&&& &&&&& &&&&& &&&&& && \\
&&& f &&&& b &&&&& &&&&& &&&&& &&&&& && \\
&&&& d & c &&&&& &&&&& &&&&& &&&&& &&&& \\
&&&& e && c &&&&& &&&&& &&&&& &&&&& &&& \\
&&&&& &&&&& && f & e && b & c && a &&&&& &&&&& & \\
&&&&& &&&&& & f && d & b &&& a && c &&&&& &&&&& \\
&&&&& &&&&& & e & d && c &&&& c && a & b &&&&& &&& \\
&&&&& &&&& c & b &&&&&&& e & d && f &&&&& &&&& \\
&&&&& &&& c && a &&&& e & d &&&&&& f &&&&& &&& \\
&&&&& &&& b & a &&&&& f && d && f & e &&&&& &&&&& \\
&&&&& &&& d & e & f & a & b & c &&&&& &&&&& &&&&& & \\
&&&&& &&&&& &&&&& &&&&& && d & e & f & a & b & c &&
\end{array}\right)
$$

Blank entries are taken to be zero (there are two blank rows at the top and two blank columns on the right). These matrices, for different values of $a, b, c, d, e, f \in k$, all commute and square to zero. Furthermore, as long as a, b, c, d, e, f are not all zero, the matrix has rank exactly 14. So the kernel modulo the image is a two-dimensional vector space. This means that we have assigned to each point $(a : b : c : d : e : f) \in \mathbb{P}^5$ a vector space of dimension two. This gives us a rank two *algebraic vector bundle* on projective 5-space, which happens to be the indecomposable *Tango bundle*.

On the other hand, if we add the identity to the six matrices obtained by setting one of the variables equal to 1 and the rest equal to 0, we obtain six commuting matrices which square to the identity. In other words, we have a kE-module, where $E \cong (\mathbb{Z}/2)^6$ is an elementary abelian 2-group of rank six. This module has *constant Jordan type* $[2]^{14}[1]^2$, meaning that the matrices above all have the same Jordan canonical form with 14 blocks of length two and two blocks of length one.

This illustrates the connection between modules of constant Jordan type for elementary abelian p-groups and vector bundles on projective space in characteristic p investigated in this book. Because the people studying these two subjects are almost disjoint, I have tried to include plenty of introductory material. The reader should feel free to skip this if appropriate.

So what are modules of constant Jordan type? Among all modules, they are analogous to the vector bundles among the sheaves.

As Dade put it in [97], “There are just too many modules over p-groups!” More explicitly, the group algebra of a finite p-group in characteristic p usually has *wild representation type*, as we explain in Section 1.2. For a more general finite group, the Sylow p-subgroup controls the representation type. It follows that in general we do not hope to classify all the finite-dimensional indecomposable representations of a finite group.

The theory of varieties for modules was developed by Carlson and others [24, 73, 74, 75] as a way of getting at module structure without making such a classification. Many aspects of this theory are controlled by the elementary abelian p-subgroups. So it makes sense to study modules for an elementary abelian p-group as a subject in its own right. We concentrate on modules of constant Jordan type. For these modules the variety gives essentially no information, so the theory supplements the now well-established variety theory. Modules of constant Jordan type are still wild whenever the representation type of the group is wild, as we show in Section 5.5. These modules are much more rigidly behaved than the general module, but the theory is nonetheless surprisingly rich.

The formal definition of constant Jordan type is as follows. Let $E = \langle g_1, \dots, g_r\rangle \cong (\mathbb{Z}/p)^r$ be an elementary abelian p-group and let k be an algebraically closed field of characteristic p. We set

$$X_i = g_i - 1 \in kE$$

for $1 \le i \le r$, so that $X_i^p = 0$. If $\alpha = (\lambda_1, \dots, \lambda_r) \in \mathbb{A}^r(k)$, we define

$$X_\alpha = \lambda_1 X_1 + \cdots + \lambda_r X_r \in kE.$$

If $\alpha \ne 0$ then $g_\alpha = 1 + X_\alpha$ is a unit of order p in kE.

A *cyclic shifted subgroup* of E is a subgroup of the group algebra kE of the form $E_\alpha = \langle g_\alpha\rangle$ for $0 \ne \alpha \in \mathbb{A}^r(k)$. A finitely generated kE-module is said to have *constant Jordan type* if the Jordan canonical form of X_α on M is independent of α for $0 \ne \alpha \in \mathbb{A}^r(k)$.

We are far from understanding what Jordan types occur for a module M of constant Jordan type. Of course if $r = 1$ then the problem is trivial. So let us assume that $r \geq 2$. We write $[a_1]\dots[a_t]$ for a Jordan type with blocks of lengths $a_1, \dots, a_t$, each of which is an integer between 1 and p. We often wish to ignore Jordan blocks of length p, and the *stable Jordan type* is the same list with the length p blocks omitted. The first important theorem in the subject is Dade's lemma, from his 1978 paper [98], which states that if the stable Jordan type is empty, in other words if M has constant Jordan type $[p]^n$ for some $n \geq 0$, then M is projective. In particular, n is divisible by p^{r-1}. Using this, it is not hard to show that a module M of stable constant Jordan type [1] or $[p-1]$ is *endotrivial*, in the sense that $M \otimes_k M^*$ is trivial plus projective. Dade's classification of endotrivial modules for an elementary abelian p-group then implies that M is isomorphic to $\Omega^n(k)$ plus a projective for some $n \in \mathbb{Z}$.

In the paper of Carlson, Friedlander and Pevtsova [81], it is conjectured that for $r \geq 2$ and $p \geq 5$ there is no module of stable constant Jordan type [2]. In other words, there is no module with the property that every X_α acts with Jordan blocks all of length p except for a single block of length two. In Section 5.12 we prove the more general statement that for $r \geq 2$ and $2 \leq a \leq p-2$ there is no kE-module of stable constant Jordan type $[a]$. This completes the analysis of modules of constant Jordan type with one non-projective Jordan block. For larger stable Jordan types, our knowledge is much more limited. The following conjectures appear in Sections 5.13 and 5.15. We continue to assume that E has rank $r \geq 2$.

Conjecture (Rickard) If a kE-module of constant Jordan type has no Jordan blocks of length j then the total number of Jordan blocks of length larger than j (including the blocks of length p) is divisible by p.

In Section 5.13 we prove the cases $j = 1$ and $j = p - 1$ of Rickard's conjecture. The proof involves the notion of *generic kernel* for modules over rank two elementary abelian groups, developed by Carlson, Friedlander and Suslin [82].

Conjecture (Suslin) If a kE-module of constant Jordan type has Jordan blocks of length j then it also has to have Jordan blocks of length either $j + 1$ or $j - 1$. In other words, there are no isolated lengths.

Conjecture (Carlson, Friedlander and Pevtsova) If M is a kE-module of stable constant Jordan type $[2][1]^j$ then $j \geq r - 1$.

In Section 10.4 we prove a weak version of the conjecture of Carlson, Friedlander and Pevtsova, on modules of stable constant Jordan type $[2][1]^j$. Namely we prove that $j \geq r-2$ if p is large enough. The proof of this uses the theory of Chern classes for vector bundles on projective space, a subject which we discuss in detail in Chapter 7.

When we talk of vector bundles, we are referring to *algebraic vector bundles*, namely locally free sheaves of modules over the structure sheaf. In the case of projective space $\mathbb{P}^{r-1}$, the only rank one vector bundles (line bundles) are twists of the structure sheaf $\mathcal{O}(a)$. There are plenty of indecomposable vector bundles of every rank at least $r-2$ if $r \geq 3$, but very little is known about vector bundles whose rank s is in the range $2 \leq s \leq r-3$. The only values of r and s in this range for which we know of indecomposable vector bundles are $r = 5$, $s = 2$ (Horrocks–Mumford), $r = 6$, $s = 3$ (Horrocks), and in characteristic two $r = 6$, $s = 2$ (Tango). A vector bundle in this range with other values of r and s will be referred to as a *new low rank vector bundle* on projective space.

Part of the point of Chapters 6 and 7 is to give an account of the theory of vector bundles on projective space, leading quickly and efficiently to the definition of Chern classes and a proof of the Hirzebruch–Riemann–Roch theorem in this case. This will greatly facilitate the discussion of restrictions on Jordan type coming from Chern classes.

If M is a kE-module of constant Jordan type, then we associate to M vector bundles $\mathcal{F}_1(M), \dots, \mathcal{F}_p(M)$ on projective space $\mathbb{P}^{r-1}$ in such a way that the rank of $\mathcal{F}_i(M)$ is equal to the number of Jordan blocks of length i on a cyclic shifted subgroup of kE. The way that projective space enters the game is that points in $\mathbb{P}^{r-1}$ correspond to cyclic shifted subgroups of kE (up to a scalar). In some sense, $\mathcal{F}_i(M)$ associates to each point of projective space the socle of the sum of the Jordan blocks of length i of the restriction of M to the corresponding cyclic shifted subgroup. The twists $\mathcal{F}_i(M), \mathcal{F}_i(M)(1), \dots, \mathcal{F}_i(M)(i-1)$ associate to each point on projective space the successive socle layers of the length i blocks of the restriction.

At this point, an interesting question arises. What vector bundles occur this way? The answer to this question is quite different for $p = 2$ and p odd, as we shall see in Section 8.9. For $p = 2$, given any vector bundle $\mathcal{F}$ of rank s on $\mathbb{P}^{r-1}$, there exists a kE-module M of stable constant Jordan type $[1]^s$ such that $\mathcal{F}_1(M) \cong \mathcal{F}$. The same construction with p odd only shows that given $\mathcal{F}$, there is a kE-module M of stable constant Jordan type $[1]^s$ such that $\mathcal{F}_1(M) \cong F^*(\mathcal{F})$, the pullback of $\mathcal{F}$ through the Frobenius map F on $\mathbb{P}^{r-1}$. And indeed, it turns out that without pulling back through the Frobenius map, there are restrictions coming from Chern classes. In Section 10.3 we show that if M has stable constant Jordan type $[1]^s$ then the Chern numbers $c_1(\mathcal{F}_1(M)), \dots, c_{p-2}(\mathcal{F}_1(M))$ are divisible by p.

Further congruences on Chern numbers come from the Hirzebruch–Riemann–Roch theorem. For example, in Section 10.8 we prove the following. Let $E \cong (\mathbb{Z}/2)^r$. If M is a kE-module of constant Jordan type $[2]^n[1]^m$ and n is not congruent to one of the integers $0, -1, \dots, -m$ modulo 2^{r-1} then $\mathcal{F}_1(M)$ is a vector bundle on $\mathbb{P}^{r-1}$ which is not a sum of line bundles. In particular, if $r > 6$ and $m \leq r-3$ then $\mathcal{F}_1(M)$ is a new low rank vector bundle on projective space. The Tango bundle example from the beginning of the introduction shows why the restriction $r > 6$ is necessary.

Finally, the last chapter is a bit more speculative. We investigate the general question of how to construct small modules with interesting properties. The constructions are basically the same as those used to construct modules of constant Jordan type, but we see that the applicability of the methods is much wider. Again, $p = 2$ behaves quite differently from p odd. So, for example, we shall see that a module of Loewy length p can have an arbitrary hypersurface as its variety if $p = 2$ but only finite unions of hyperplanes can be realised when p is odd.