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David J. Benson
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Modular Representations and Elementary Abelian Groups

1.1 Introduction

This chapter introduces background material on the relevant aspects of modular representation theory of finite elementary abelian p-groups. We work over an algebraically closed field k of characteristic p, and we only consider finitely generated modules, even when some statements are true more generally.

Since the group algebra of an elementary abelian 2-group is just an exterior algebra, many of the methods we describe apply equally well to modules over exterior algebras in any characteristic. So to some extent we develop the theory of exterior algebras in parallel to that of elementary abelian p-groups.

In the representation theory of elementary abelian p-groups, we often have to describe separately the cases $p = 2$ and p odd. In the case of exterior algebras, usually no such separation of cases is necessary.

1.2 Representation Type

The *trichotomy theorem* (Drozd [107], Crawley–Boevey [96]) partitions finite-dimensional algebras over an algebraically closed field into three mutually disjoint classes:

(i) Finite representation type: in this case there are only a finite number of isomorphism classes of finitely generated indecomposable modules.

(ii) Tame representation type: in this case there are infinitely many isomorphism classes of finitely generated indecomposables, but in any given dimension they come in one-parameter families, with finitely many exceptions. For algebras of tame representation type, one usually hopes to write down a complete classification of the finitely generated indecomposable modules.

(iii) Wild representation type: an algebra A has wild representation type if there is a finitely generated A-$k\langle X, Y\rangle$-bimodule B, free as a right $k\langle X, Y\rangle$-module, such that the functor $B \otimes_{k\langle X,Y\rangle} -$ from finite-dimensional $k\langle X, Y\rangle$-modules to finite-dimensional A-modules preserves indecomposability and reflects isomorphisms.

Broadly speaking, this means that a classification of finite-dimensional indecomposable A-modules would entail the classification of pairs of square matrices under simultaneous conjugation. This problem is thought to be unsolvable.

Group algebras of finite groups are almost always of wild representation type.

Theorem 1.2.1 (Bondarenko and Drozd [57]; see also Ringel [220]) *Let G be a finite group and k have characteristic p.*

(i) *If the Sylow p-subgroups of G are cyclic then kG has finite representation type.*

(ii) *If $p = 2$ and the Sylow p-subgroups of G are dihedral, semidihedral or generalised quaternion, then kG has tame representation type.*

(iii) *In all other cases kG has wild representation type.*

Looking in particular at an elementary abelian p-group E, this says that finite representation type happens when E has rank one, tame representation type happens only for $(\mathbb{Z}/2)^2$, and otherwise the representation type is wild.

In the rank one case, the classification follows easily from the theory of the Jordan canonical form. There is one indecomposable module for each size of Jordan block between 1 and p. In the case of $(\mathbb{Z}/2)^2$, the classification follows from Kronecker's classification of matrix pencils, see for example Section 4.3 of [39].

Next we discuss exterior algebras. Let $\Lambda = \Lambda(X_1, \dots, X_r)$ be an exterior algebra over a field k on generators $X_1, \dots, X_r$. We can either consider this as an ungraded algebra, and look at ungraded modules over it, or we can consider it as a graded algebra, and look at graded modules. In the latter case, we put the generators X_i in degree one. The relations satisfied by the X_i are $X_i^2 = 0$ $(1 \le i \le r)$ and $X_i X_j = -X_j X_i$ $(1 \le i < j \le r)$.

Theorem 1.2.2 *If $\Lambda = \Lambda(X_1, \dots, X_r)$ is an exterior algebra over a field k on generators $X_1, \dots, X_r$, then the representation type of Λ either as an ungraded or as a graded algebra is as follows:*

(i) *finite if $r = 1$,*
(ii) *tame if $r = 2$, and*
(iii) *wild if $r \ge 3$.*

In case (i), if Λ is graded, we really mean that there are finitely many isomorphism classes of indecomposable modules up to shift in degree. In fact, there are only two isomorphism classes of indecomposables, of dimensions one and two, and shifts of them in the graded situation. If $r = 2$ then again the classification of the indecomposables follows from Kronecker's classification. For $r \ge 3$, wildness follows for example from Ringel [220], or from the wildness of the quiver $\bullet \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \bullet$.

Given that we usually do not hope to classify the indecomposable kE-modules or the indecomposable Λ-modules, there are several approaches to making progress. One approach is to make classifications that are less refined. For example, the classification of thick subcategories of the stable module category is achieved in [49]. Another is to find general properties of modules, short of their classification. Examples of theorems in this direction are Dade's lemma 1.9.5 and Carlson's theory of rank varieties, discussed in Section 1.9, both of which we shall make use of in these notes. A third way to make progress is to restrict the class of modules under consideration. That is the approach taken in these notes. We shall concentrate largely on modules of constant Jordan type. There are many other interesting subclasses of the class of all finitely generated kE-modules, and we are in no way suggesting that this is the most important class.

1.3 Shifted Subgroups

We begin with a discussion of shifted subgroups of an elementary abelian p-group E. These are certain subgroups of the group algebra kE.

The notation that we use throughout this book is as follows. A finite group E is said to be an *elementary abelian p-group* if it is abelian and has exponent p. Equivalently, E is isomorphic to a direct product of r copies of a cyclic group of order p for some $r \geq 0$:

$$E = \langle g_1, \ldots, g_r \rangle \cong (\mathbb{Z}/p)^r$$

where $[g_i, g_j] = 1$ for $1 \leq i, j \leq r$ and $g_i^p = 1$ for $1 \leq i \leq r$.

We begin with the definition of shifted subgroup. Let

$$X_i = g_i - 1 \in kE$$

for $1 \leq i \leq r$. Since we are in characteristic p we have

$$X_i^p = (g_i - 1)^p = g_i^p - 1^p = 0$$

in kE. The images of $X_1, \ldots, X_r$ form a basis for $J(kE)/J^2(kE)$, where $J(kE)$ denotes the Jacobson radical of kE. If $\alpha = (\lambda_1, \ldots, \lambda_r)$ is an element of affine space $\mathbb{A}^r(k)$, we define

$$X_\alpha = \lambda_1 X_1 + \cdots + \lambda_r X_r \in kE.$$

This is an element of $J(kE)$ which again satisfies $X_\alpha^p = 0$. It follows that $(1 + X_\alpha)^p = 1$. So if $\alpha \neq 0$ then $g_\alpha = 1 + X_\alpha$ is an element of order p in the group of units of the group algebra kE.

Definition 1.3.1 A *cyclic shifted subgroup* of E is a subgroup E_α of the group of units of kE generated by such an element $g_\alpha = 1 + X_\alpha$ with $\alpha \neq 0$. More generally, a *shifted subgroup* is a subgroup E' of the group of units in kE generated by elements $g_{\alpha_1}, \ldots, g_{\alpha_s}$ where $\alpha_1, \ldots, \alpha_s$ are linearly independent elements of

$\mathbb{A}^r(k)$. This linear independence condition is equivalent to the statement that the induced map $\rho\colon kE' \to kE$ is injective. If this is the case then kE is free as a module over the image of ρ.

A *maximal shifted subgroup* is a shifted subgroup E' of the same rank as E. The induced map $\rho\colon kE' \to kE$ is then an isomorphism. Choosing a basis of E' allows us to identify it with E and obtain an automorphism ρ of kE.

This notion of a cyclic shifted subgroup is dependent on the choice of generators for E. For a different choice of generators, if we make the corresponding linear transformation on $\mathbb{A}^r(k)$, the resulting element X_α with respect to the new basis differs from the old one by an element of $J^2(kE)$.

Let us illustrate this by example. If $E = \langle g_1, g_2\rangle \cong (\mathbb{Z}/p)^2$ and $X_1 = g_1 - 1$, $X_2 = g_2 - 1$, then a multiplicative basis change on E using the matrix

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

gives new generators $E = \langle g_1 g_2, g_2\rangle$. Now we have

$$g_1 g_2 - 1 = X_1 + X_2 + X_1 X_2.$$

So modulo the square of the radical, we have the additive base change on X_1 and X_2 given by the same matrix. However, the linear span of $g_1g_2 - 1$ and $g_2 - 1$ is not the same as the linear span of X_1 and X_2, so the definition of a shifted subgroup is basis dependent. In Chapter 4, we shall make a systematic study of the effect of adding an element of $J^2(kE)$.

The following is immediate from the definitions.

Theorem 1.3.2 *The group algebra kE is a truncated polynomial ring:*

$$kE = k[X_1, \dots, X_r]/(X_1^p, \dots, X_r^p).$$

We define the *rank variety* of E to be the affine space $V_E^{\mathrm{rk}} \cong \mathbb{A}^r(k)$ obtained by taking the linear span in $J(kE)$ of the elements

$$X_1 = g_1 - 1, \dots, X_r = g_r - 1.$$

Thus V_E^{rk} is a complement in $J(kE)$ of $J^2(kE)$. This is the home for the theory of rank varieties, as described in Section 1.9.

Warning 1.3.3 It is easy fall into the trap of thinking that the definitions

$$\begin{aligned} X_i &= \log(g_i) = (g_i - 1) - \tfrac{1}{2}(g_i - 1)^2 + \cdots \\ X_\alpha &= \lambda_1 X_1 + \cdots + \lambda_r X_r \\ g_\alpha &= \exp(X_\alpha) = 1 + X_\alpha + \tfrac{1}{2!}X_\alpha^2 + \cdots \end{aligned}$$

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give a more invariant definition of shifted subgroups. The problem is that it is *not true* that

$$\log(gh) = \log(g) + \log(h).$$

For example, in characteristic two we have $\log(g) = 1+g$, $\log(g)+\log(h) = g+h$ and $\log(gh) = 1 + gh$.

The existence of an invariant definition of shifted subgroups would imply the existence of a non-zero $GL(r, \mathbb{F}_p)$-invariant homomorphism $\phi\colon E \to J(kE)$. Such a homomorphism exists if and only if E is either cyclic or isomorphic to $(\mathbb{Z}/2)^2$.

To see this, we use the following argument, due to Serge Bouc. Let ϕ be such a homomorphism and write

$$\phi(g) = \sum_{\substack{h\in E\\ h\neq 1}} \phi_h(g)(h-1).$$

Invariance amounts to the statement that for $\alpha \in GL(r, \mathbb{F}_p)$ we have $\phi_h(\alpha(g)) = \phi_{\alpha^{-1}(h)}(g)$. The group $GL(r, \mathbb{F}_p)$ is transitive on pairs of elements g, h such that g is not in $\langle h\rangle$. So $\phi_h(g)$ is a constant a, independent of h and g, provided $g \notin \langle h\rangle$. If E is neither cyclic nor isomorphic to $(\mathbb{Z}/2)^2$ then we can find linearly independent elements g and h such that $\langle g\rangle$, $\langle h\rangle$ and $\langle gh\rangle$ do not exhaust E. Then the statement that $\phi(gh) = \phi(g) + \phi(h)$ implies first that $2a = a$, so that $a = 0$, and then that $\phi = 0$ since $\phi(gh)$, $\phi(g)$ and $\phi(h)$ have supports intersecting in the identity.

The discussion of rank varieties for exterior algebras is essentially the same. Let $\Lambda = \Lambda(X_1, \dots, X_r)$ be an exterior algebra on generators $X_1, \dots, X_r$. If $\alpha = (\lambda_1, \dots, \lambda_r) \in \mathbb{A}^r(k)$, we define

$$X_\alpha = \lambda_1 X_1 + \cdots + \lambda_r X_r \in \Lambda.$$

This is an element of $J(\Lambda)$ which satisfies $X_\alpha^2 = 0$. The *rank variety* of Λ is the affine space $V_\Lambda^{\mathrm{rk}} \cong \mathbb{A}^r(k)$ obtained by taking the linear span of $X_1, \dots, X_r$ in $J(\Lambda)$. It is a complement to $J^2(\Lambda)$ in $J(\Lambda)$. Details can be found in the paper of Aramova, Avramov and Herzog [12].

1.4 The Language of π-Points

Much of the modern literature on Jordan type for finite group schemes [80, 81, 82, 118, 124, 125, 126, 127, 128] is written in the language of π-points. It is worth making a few remarks on the translation between this and the language of cyclic shifted subgroups, in the case of an elementary abelian p-group. Understanding this section is not logically necessary for the rest of the book, but will help reconcile it with the rest of the literature.

Giving a finite group scheme G over k is equivalent to giving a finite-dimensional cocommutative Hopf algebra kG over k. For example, if G is a finite group then kG is just its group algebra in the normal sense. Other examples include the restricted universal enveloping algebra of a finite-dimensional p-restricted Lie algebra. In this example the underlying variety of the group scheme has only one point, but the local ring at that point contains all the information – it is just the dual Hopf algebra.

If G is a finite group scheme then a π-point of G is defined to be a *flat* homomorphism of algebras $K[t]/(t^p) \to KG$ over some extension field K of k, which factors through some unipotent abelian subgroup scheme of KG.

To say that a homomorphism $K[t]/(t^p) \to KG$ is flat means that KG is flat as a module over $K[t]/(t^p)$, which is equivalent to each one of the following adjectives: *free, projective, injective*.

An equivalence relation is put on π-points as follows. Two π-points are said to be equivalent if for every finitely generated kG-module M, the restriction of $K \otimes_k M$ along one is free if and only if the restriction along the other is free. This condition is difficult to work with in practice, since it is expressed in terms of all finitely generated modules, but can be reduced to an easier cohomological condition. See for example Theorem 3.6 of [125].

In the case where $G = E$ is a finite elementary abelian p-group, KE is unipotent, so the second condition is automatically satisfied. The flatness condition is equivalent to the statement that the image of t lies in $J(KE)$ but not in $J^2(KE)$. The equivalence relation in this case is as follows. By Lemma 6.4 of Carlson [76], two such homomorphisms give equivalent π-points if and only if the differences between the images of t lies in $J^2(KE)$. Thus the set of equivalence classes of π-points over K is in bijection with $\mathbb{A}^r(K) \setminus \{0\}$. With respect to a given choice of generators of E, in each equivalence class there is a unique representative which is given by a cyclic shifted subgroup.

Note that an exterior algebra in characteristic not equal to two is not an example of a finite group scheme. Nonetheless, the notion of π-point makes perfect sense in this context, if defined as a flat embedding $K[t]/(t^2) \to K \otimes_k \Lambda$.

1.5 The Stable Module Category

In general, if G is a finite group the cohomology ring $H^*(G, k)$ is defined to be $\mathrm{Ext}^*_{kG}(k, k)$. By a theorem of Evens [115] and Venkov [242] (or an earlier theorem of Golod [136] in the case of finite p-groups), this is a finitely generated graded commutative algebra over k.

The *module category* $\mathsf{mod}(kG)$ is the category of finitely generated kG-modules and module homomorphisms. The *stable module category* $\mathsf{stmod}(kG)$ has the same objects as $\mathsf{mod}(kG)$ but its arrows are given by

$$\underline{\mathrm{Hom}}_{kG}(M, N) = \mathrm{Hom}_{kG}(M, N)/P\mathrm{Hom}_{kG}(M, N),$$

where $P\mathrm{Hom}_{kG}(M, N)$ is the linear subspace of $\mathrm{Hom}_{kG}(M, N)$ consisting of homomorphisms that factor through some projective kG-module. Whereas $\mathsf{mod}(kG)$ is an abelian category, $\mathsf{stmod}(kG)$ is a triangulated category.

If M is a finitely generated kG-module, we define $\Omega(M)$ to be the kernel of the projective cover of M and $\Omega^{-1}(M)$ to be the cokernel of the injective hull of M. It follows from the fact that kG is self-injective that $\Omega(\Omega^{-1}(M))$ and $\Omega^{-1}(\Omega(M))$ are naturally isomorphic to M in the stable module category $\mathsf{stmod}(kG)$. If M has no projective summands, this implies that they are isomorphic to M in the module category $\mathsf{mod}(kG)$, but the isomorphism is not natural. If $n > 0$ we have

$$\mathrm{Ext}^n_{kG}(M, N) \cong \underline{\mathrm{Hom}}_{kG}(\Omega^n(M), N), \tag{1.1}$$

and in particular since $P\mathrm{Hom}_{kG}(\Omega^n(k), k) = 0$ we have

$$H^n(G, k) \cong \mathrm{Ext}^n_{kG}(k, k) \cong \mathrm{Hom}_{kG}(\Omega^n(k), k).$$

If $\zeta \in H^n(G, k)$, we write $\hat{\zeta}$ for the corresponding homomorphism $\Omega^n(k) \to k$. If $\zeta \neq 0$ then $\hat{\zeta}$ is surjective, and the *Carlson module* L_ζ is defined to be the kernel of $\hat{\zeta}$. We shall study these modules further in Sections 1.10 and 12.4.

Tate cohomology is defined by extending (1.1) from positive degrees to all degrees:

$$\widehat{\mathrm{Ext}}^n_{kG}(M, N) \cong \underline{\mathrm{Hom}}_{kG}(\Omega^n(M), N) \qquad (n \in \mathbb{Z}).$$

The Tate cohomology ring of G is defined to be

$$\hat{H}^*(G, k) = \widehat{\mathrm{Ext}}^*_{kG}(k, k)$$

with multiplication given by shifting and composing. This makes it into a graded commutative k-algebra with the ordinary cohomology $H^*(G, k)$ as the subalgebra of non-negative degree elements. The k-algebra $\hat{H}^*(G, k)$ is finitely generated if and only if it is periodic, which in turn happens if and only if all abelian p-subgroups of G are cyclic.

Tate duality (see for example Cartan and Eilenberg [83], section XII.6) states that for $n \in \mathbb{Z}$ there are natural vector space isomorphisms

$$(\hat{H}^{-n-1}(G, M))^* \cong \widehat{\mathrm{Ext}}^n_{kG}(M, k).$$

In particular, $\hat{H}^{-n-1}(G, k)$ is the vector space dual of $\hat{H}^n(G, k)$.

If H is a subgroup of G, then transfer in Tate cohomology from H to G is Tate dual to restriction from G to H.

Exactly the same definitions make sense in the case of an exterior algebra. If $\Lambda = \Lambda(X_1, \dots, X_r)$ is an exterior algebra over k then the cohomology ring $H^*(\Lambda, k) = \mathrm{Ext}^*_\Lambda(k, k)$ is a polynomial ring $k[Y_1, \dots, Y_r]$. If Λ is regarded here as an *ungraded* algebra, then the Y_i are in cohomological degree one. So if k does not have characteristic two and $r > 1$ then $H^*(\Lambda, k)$ is not a graded commutative ring, because the Y_i do not commute with each other in the graded sense. On the other hand, if Λ is regarded as a *graded* algebra with the X_i in degree one, then

$H^*(\Lambda, k)$ is bigraded, with the Y_i in degree $(1, 1)$. The total degree is two, so the cohomology ring is graded commutative in this case.

Tate cohomology and Tate duality for exterior algebras works in the same way as for group algebras.

1.6 The Derived Category

If $\mathcal{A}$ is an abelian category, we write $\mathsf{D}^b(\mathcal{A})$ for the *bounded derived category* of $\mathcal{A}$. Its objects are the bounded chain complexes of objects and arrows in $\mathcal{A}$. The morphisms are the homotopy classes of degree preserving chain maps, with the quasi-isomorphisms inverted; a *quasi-isomorphism* is a morphism that induces an isomorphism in homology. Thus a morphism $\mathbf{C} \to \mathbf{D}$ in $\mathsf{D}^b(\mathcal{A})$ is given by an equivalence class of diagrams of chain complexes

$$\begin{tikzcd} & \mathbf{C}' \arrow[dl, "\mathrm{qi}"'] \arrow[dr] & \\ \mathbf{C} & & \mathbf{D} \end{tikzcd}$$

where "qi" denotes a quasi-isomorphism. Two such diagrams $\mathbf{C} \xleftarrow{\mathrm{qi}} \mathbf{C}' \to \mathbf{D}$ and $\mathbf{C} \xleftarrow{\mathrm{qi}} \mathbf{C}'' \to \mathbf{D}$ are equivalent if there is a third $\mathbf{C} \xleftarrow{\mathrm{qi}} \mathbf{C}''' \to \mathbf{D}$ that maps to both of them to make a commutative diagram

$$\begin{tikzcd} & & \mathbf{C}''' \arrow[dl, "\mathrm{qi}"'] \arrow[dr] & & \\ & \mathbf{C}' \arrow[dl, "\mathrm{qi}"'] \arrow[drrr] & & \mathbf{C}'' \arrow[dlll, "\mathrm{qi}"] \arrow[dr] & \\ \mathbf{C} & & & & \mathbf{D}. \end{tikzcd}$$

To compose arrows we form a pullback:

$$\begin{tikzcd} & & \mathbf{C}'' \arrow[dl, dotted, "\mathrm{qi}"'] \arrow[dr, dotted] & & \\ & \mathbf{C}' \arrow[dl, "\mathrm{qi}"'] \arrow[dr] & & \mathbf{D}' \arrow[dl, "\mathrm{qi}"] \arrow[dr] & \\ \mathbf{C} & & \mathbf{D} & & \mathbf{E}. \end{tikzcd}$$

The distinguished triangles in $\mathsf{D}^b(\mathcal{A})$ are the triangles isomorphic to

$$\mathbf{C} \xrightarrow{f} \mathbf{D} \to M_f \to \mathbf{C}[1]$$

where M_f is the mapping cone of f.

The derived categories we shall be interested in are $\mathsf{D}^b(kG) = \mathsf{D}^b(\mathsf{mod}(kG))$, the derived category of kG-modules, and $\mathsf{D}^b(\mathsf{Coh}(X))$, the derived category of coherent $\mathcal{O}_V$-modules for a variety V.

Next we describe the construction of a functor

$$\mathcal{R}\colon \mathsf{D}^b(kG) \to \mathsf{stmod}(kG).$$

This functor appeared in the late 1980s in the work of several people, see Buchweitz [66], Keller and Vossieck [176], Rickard [219, Theorem 2.1]. We shall use this construction in Section 8.9 as part of the proof of the realisation theorem.

Definition 1.6.1 A *perfect complex* is an object in $\mathsf{D}^b(kG)$ which is isomorphic to a bounded complex of projective modules. We write $\mathsf{perf}(kG)$ for the full subcategory of $\mathsf{D}^b(kG)$ whose objects are the perfect complexes.

The quotient category $\mathsf{D}^b(kG)/\mathsf{perf}(kG)$ has the same objects as $\mathsf{D}^b(kG)$. The arrows are formed by inverting arrows whose mapping cone is in $\mathsf{perf}(kG)$.

Theorem 1.6.2 *The quotient category* $\mathsf{D}^b(kG)/\mathsf{perf}(kG)$ *is equivalent to the stable module category* $\mathsf{stmod}(kG)$.

Proof (sketch). We begin with the functor $\mathsf{mod}(kG) \to \mathsf{D}^b(kG)$ which takes a module M to the complex which consists of M in degree zero and the zero module in all other degrees. The composite of this with the quotient functor

$$\mathsf{D}^b(\mathsf{mod}(kG)) \to \mathsf{D}^b(kG)/\mathsf{perf}(kG)$$

takes projective modules to zero, and so it factors through $\mathsf{mod}(kG) \to \mathsf{stmod}(kG)$ to give a well-defined functor

$$\mathsf{stmod}(kG) \to \mathsf{D}^b(kG)/\mathsf{perf}(kG).$$

One can check that this functor takes distinguished triangles to distinguished triangles. It is also full, and takes non-zero objects to non-zero objects. To prove that it is an equivalence, it therefore remains to prove that every object in $\mathsf{D}^b(kG)/\mathsf{perf}(kG)$ is isomorphic to an object in the image of this functor.

Given a bounded chain complex of finitely generated kG-modules $\mathbf{C}$, we can resolve it to give a quasi-isomorphism

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] & P_s \arrow[r] \arrow[d] & \cdots \arrow[r] & P_t \arrow[r] \arrow[d] & 0 \\ & 0 \arrow[r] & M_s \arrow[r] & \cdots \arrow[r] & M_t \arrow[r] & 0 \end{tikzcd}$$

where the P_i are projective modules, and without loss of generality we can take $s > 0$. Now take an injective resolution of K. Then there is a quasi-isomorphism

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[d] & Q_s \arrow[r] \arrow[d] & \cdots \arrow[r] & Q_1 \arrow[r] \arrow[d] & Q_0 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & \Omega^{-s-1}(K) \arrow[r] & 0. \end{tikzcd} \tag{1.2}$$

Then modulo $\mathsf{perf}(kG)$ the original complex is isomorphic to K in degree $s+1$, which is in turn isomorphic to the top row and then the bottom row of (1.2). The latter is in the image of $\mathsf{stmod}(kG)$. □

Definition 1.6.3 We write $\mathcal{R}$ for the functor

$$\mathcal{R}\colon \mathsf{D}^b(kG) \to \mathsf{stmod}(kG)$$

whose existence is guaranteed by the theorem. This functor takes distinguished triangles in $\mathsf{D}^b(kG)$ to distinguished triangles in $\mathsf{stmod}(kG)$.

The proof of the theorem shows that the image of a complex under this functor is obtained by resolving, taking the kernel, and then shifting it back to degree zero. It is possible, but tedious to check directly that this process is functorial.

It follows from the theorem that an object in $\mathsf{D}^b(kG)$ goes to zero in $\mathsf{stmod}(kG)$ if and only if it is a perfect complex.

In the case of the exterior algebra Λ, there are two versions of the derived category and of the stable module category, according to whether we are talking about ungraded modules or graded modules. In both cases, the discussion of the derived category and its relationship with the stable module category works in the same way as above. More generally, it works for any finite-dimensional ungraded or graded self-injective algebra.

1.7 Singularity Categories

As a generalisation of the equivalence $\mathsf{stmod}(kG) \simeq \mathsf{D}^b(kG)/\mathsf{perf}(kG)$ of Theorem 1.6.2, we have the following definition, which will become important in Chapter 11.

Definition 1.7.1 Let R be a ring. Then the *singularity category* of R is the Verdier quotient

$$\mathsf{D}_{\mathrm{sg}}(R) = \mathsf{D}^b(R)/\mathsf{perf}(R).$$

Likewise, if R is a graded ring, we denote by $\mathsf{D}^b(R)$ the bounded derived category of finitely generated graded R-modules and $\mathsf{D}_{\mathrm{sg}}(R)$ the quotient by the perfect complexes of graded modules.

Warning 1.7.2 In commutative algebra, this definition is much better behaved for Gorenstein rings than for more general commutative Noetherian rings. For a Gorenstein ring, the singularity category is equivalent to the stable category of maximal Cohen–Macaulay modules (Buchweitz [66]), but the following example is typical of the behaviour for non-Gorenstein rings.