QUANTUM INFORMATION AND QUANTUM OPTICS
WITH SUPERCONDUCTING CIRCUITS

Superconducting quantum circuits are among the most promising solutions for the development of scalable quantum computers. Built with sizes that range from microns to tens of meters using superconducting fabrication techniques and microwave technology, superconducting circuits demonstrate distinctive quantum properties such as superposition and entanglement at cryogenic temperatures. This book provides a comprehensive and self-contained introduction to the world of superconducting quantum circuits and how they are used in current quantum technology. Beginning with a description of their basic superconducting properties, the author then explores their use in quantum systems, showing how they can emulate individual photons and atoms and ultimately behave as qubits within highly connected quantum systems. Particular attention is paid to cutting-edge applications of these superconducting circuits in quantum computing and quantum simulation. Written for graduate students and junior researchers, this accessible text includes numerous homework problems and worked examples.

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QUANTUM INFORMATION AND QUANTUM OPTICS WITH SUPERCONDUCTING CIRCUITS

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Contents

List of Figures xi
List of Tables xiii
Notation xiv

1 Introduction 1
1.1 The Book 4
1.2 Acknowledgments 6

2 Quantum Mechanics 7
2.1 Canonical Quantization 7
2.1.1 Hamiltonian Equations 9
2.1.2 Quantum Observables 10
2.1.3 Unitary Evolution 12
2.2 Two-Level Systems 13
2.3 Density Matrices 14
2.4 Measurements 16

3 Superconductivity 19
3.1 Microscopic Model 22
3.2 Macroscopic Quantum Model 24
3.3 Superfluid Current 25
3.4 Superconducting Phase 26
3.5 Gauge-Invariant Phase 27
3.6 Fluxoid Quantization 29
3.7 Josephson Junctions 30

4 Quantum Circuit Theory 34
4.1 Introduction 34
4.1.1 What Makes a Circuit Quantum 34
# Contents

4.1.2 How Do We Work with Quantum Circuits? 38

4.2 Circuit Elements 40
  4.2.1 Capacitor 41
  4.2.2 Inductor 42
  4.2.3 Josephson Junctions or “Nonlinear Inductors” 43
  4.2.4 Other Elements 44

4.3 Quantization Procedure 44

4.4 LC Resonator 49

4.5 Transmission Line 50

4.6 Charge and Transmon Qubits 52

4.7 SQUIDs 53
  4.7.1 rf-SQUID 53
  4.7.2 dc-SQUID 55

4.8 Three-Junction Flux Qubit 57

4.9 Number-Phase Representation 59

Exercises 61

5 Microwave Photons 63
  5.1 LC Resonator 63
    5.1.1 Energy Quantization and Photons 63
    5.1.2 Hamiltonian Diagonalization 65
    5.1.3 Phase Space Dynamics 67
    5.1.4 Are There Real Photons? 68

  5.2 Transmission Lines or Waveguides 69
    5.2.1 Periodic Boundary Conditions 70
    5.2.2 $\lambda/2$ and $\lambda/2$ Microwave Cavities 71
    5.2.3 Tunable Cavities 73

  5.3 Three-Dimensional Cavities and Waveguides 75

  5.4 Photon States 76
    5.4.1 Fock States 76
    5.4.2 Thermal States 77
    5.4.3 Coherent States 78
    5.4.4 Schrödinger Cat States 79
    5.4.5 Single-, Two-, and Multimode Squeezed States 80
    5.4.6 Wigner Functions and Gaussian States 82

  5.5 Gaussian Control of Microwave Photons 85
    5.5.1 Coherent Drivings and Displacement Operations 85
    5.5.2 Coupling to an Environment 87
    5.5.3 Cavity Spectroscopy 90
    5.5.4 Losses and Heating 92
Contents

5.5.5 Beam Splitters and Circulators 93
5.5.6 Amplification 95
5.5.7 Photon Quadrature Measurements 99
5.6 Conclusion 103
Exercises 103

6 Superconducting Qubits 106
6.1 What Is a Qubit? 106
  6.1.1 From Logical to Physical Qubits 106
  6.1.2 Qubit Hamiltonian 110
  6.1.3 Interaction Picture 111
  6.1.4 Single-Qubit Gates 112
  6.1.5 Decoherence and Dephasing 112
  6.1.6 Relaxation and Heating 115
6.2 Charge Qubit 116
  6.2.1 Coulomb Blockade 116
  6.2.2 The Actual Superconducting Charge Qubit 117
  6.2.3 Qubit Hyperbola 120
  6.2.4 Charge Qubit History 121
6.3 Transmon Qubit 122
  6.3.1 Moving Particle Picture and Energy Bands 123
  6.3.2 Transmon as Anharmonic Oscillator 125
  6.3.3 Josephson Junctions and the Mathieu Equation 126
  6.3.4 Transmon as Qubit 128
6.4 Flux Qubit 130
  6.4.1 Frustration and Current States 130
  6.4.2 rf-SQUID Qubit 132
  6.4.3 Persistent Current Qubit 135
  6.4.4 General Operation 138
6.5 Qubit–Qubit Interactions 140
  6.5.1 Dipolar Magnetic Interaction 141
  6.5.2 Dipolar Electric Interaction 142
  6.5.3 Coupling Tunability 144
  6.5.4 Mediated Interactions and Tunable Couplers 145
6.6 Qubit Coherence 147
Exercises 149

7 Qubit–Photon Interaction 156
7.1 Qubit–Line Interaction Models 157
  7.1.1 Dipolar Interaction 157
  7.1.2 Spin-Boson Hamiltonian 159
Table of Contents

7.1.3 Spectral Function and Spin-Boson Regimes 160
7.1.4 Rotating Wave Approximation 163
7.2 Waveguide-QED 164
  7.2.1 Wigner–Weisskopf Approximation 165
  7.2.2 Input–Output Relations 166
  7.2.3 Spontaneous Emission Spectrum 167
  7.2.4 Single-Photon Scattering 169
  7.2.5 Quantum Links 172
7.3 Cavity-QED 173
  7.3.1 Quantum Rabi and Jaynes–Cummings Models 175
  7.3.2 Jaynes–Cummings Ladder 177
  7.3.3 Vacuum Rabi splitting 178
  7.3.4 Rabi Oscillations: Weak and Strong Coupling 179
  7.3.5 Ultrastrong Coupling 182
  7.3.6 Multiple Qubits 183
  7.3.7 Off-Resonant Qubits and Dispersive Coupling 184
7.4 Circuit-QED Control 185
  7.4.1 Direct Cavity Spectroscopy 185
  7.4.2 Qubit Dispersive Measurement 187
  7.4.3 Two-Tone Spectroscopy 190
  7.4.4 Single-Photon Generation 190
  7.4.5 Qubit Reset 191
  7.4.6 Cavity Fock States Superpositions 192
  7.4.7 Cavity Schrödinger Cats 193
Exercises 194

8 Quantum Computing 198
  8.1 Quantum Circuit Model 198
  8.2 Quantum Registers 202
    8.2.1 Measurements 202
    8.2.2 Qubit Reset 204
    8.2.3 Architectural Decisions 204
  8.3 Gate Toolbox 206
    8.3.1 Universal Set of Gates 206
    8.3.2 Two-Qubit Exchange Gates (iSWAP) 207
    8.3.3 Two-Qubit Tunable Frequency CZ Gate 209
    8.3.4 Two-Qubit Tunable Coupling CZ Gate 212
  8.4 Tomography and Error Characterization 213
    8.4.1 Classes of Errors 213
    8.4.2 Error Models: Completely Positive Maps 214
Contents

8.4.3 Error Quantification: Fidelity 217
8.4.4 Randomized Benchmarking 219
8.5 Fault-Tolerant Quantum Computers 221
  8.5.1 Local Errors and Global Qubits 222
  8.5.2 Passive versus Active Error Correction 223
  8.5.3 Stabilizer Codes 224
  8.5.4 Surface Code 225
  8.5.5 Fault-Tolerant Thresholds and Outlook 230
8.6 Near-Term Intermediate Scale Quantum Computers 232
  8.6.1 What Is NISQ? 232
  8.6.2 Hybrid Quantum Computers 233
  8.6.3 Quantum Volume 234
8.7 Outlook 234
Exercises 236

9 Adiabatic Quantum Computing 239
  9.1 Adiabatic Evolution 239
    9.1.1 Landau–Zener and Qubit Adiabatic Control 241
    9.1.2 The Adiabatic Theorem 243
    9.1.3 Circuit-QED Applications of Adiabatic Theorem 244
  9.2 Adiabatic Quantum Computing Model 245
    9.2.1 The Adiabatic Quantum Computing Algorithm 246
    9.2.2 Resource Accounting 247
  9.3 The Choice of Hamiltonian 248
    9.3.1 A Primer on Complexity Classes 248
    9.3.2 QUBO and NP-Complete Hamiltonian Problems 250
    9.3.3 QMA-Complete Problems 251
    9.3.4 Scaling of Resources 252
  9.4 D-Wave’s Quantum Annealer 253
    9.4.1 D-Wave’s Architecture 254
    9.4.2 Device Operation 257
    9.4.3 Performance Analysis 260
  9.5 Summary and Outlook 266
Exercises 267

Appendix A Hamiltonian Diagonalizations 270
  A.1 Tridiagonal Matrix Diagonalization 270
    A.1.1 Periodic Boundary Conditions 270
    A.1.2 Open Boundary Conditions 271
  A.2 Harmonic Chain Diagonalization 272
x

Contents

A.3 Schrieffer–Wolff Perturbation Theory 273
  A.3.1 Nondegenerate Perturbation Theory 273
  A.3.2 Degenerate Perturbation Theory 274
  A.3.3 Considerations 275

Appendix B  Open Quantum Systems 277
  B.1 Nonunitary Evolution 277
  B.2 Master Equations 278
    B.2.1 Lindblad Equation 279
    B.2.2 Linear System–Bath Coupling 280
    B.2.3 System in a Thermal Bath: Cooling and Heating 280
    B.2.4 Perturbations and Generalizations 281
    B.2.5 Strong Nonlinearity and Multilevel systems 282
  B.3 Input–Output Theory 282
    B.3.1 Memory Function 283
    B.3.2 Markovian Approximation: Decay Rate and Lamb Shift 284
    B.3.3 Input–Output Relations 284
    B.3.4 Spectroscopy 286

References 287

Index 299
Figures

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>Typical microwave measurement setup.</td>
</tr>
<tr>
<td>2.2</td>
<td>Quantum circuit for a generalized quantum measurement.</td>
</tr>
<tr>
<td>3.1</td>
<td>Josephson junction circuit and quantum model.</td>
</tr>
<tr>
<td>4.1</td>
<td>Thermal fluctuations in a quantum device.</td>
</tr>
<tr>
<td>4.2</td>
<td>Linear circuit elements and notation.</td>
</tr>
<tr>
<td>4.3</td>
<td>Circuit quantization procedure.</td>
</tr>
<tr>
<td>4.4</td>
<td>Quantization of the LC resonator equivalent circuit.</td>
</tr>
<tr>
<td>4.5</td>
<td>Quantization of the equivalent circuit for a microwave guide.</td>
</tr>
<tr>
<td>4.6</td>
<td>Equivalent circuit for the charge qubit.</td>
</tr>
<tr>
<td>4.7</td>
<td>Equivalent circuit for the rf-SQUID qubit.</td>
</tr>
<tr>
<td>4.8</td>
<td>Equivalent circuit for a dc-SQUID.</td>
</tr>
<tr>
<td>4.9</td>
<td>Washboard potential for a dc-SQUID.</td>
</tr>
<tr>
<td>4.10</td>
<td>Equivalent circuit for a three-junction flux qubit.</td>
</tr>
<tr>
<td>5.1</td>
<td>Photographies of superconducting resonators.</td>
</tr>
<tr>
<td>5.2</td>
<td>Sketch of a waveguide and equivalent circuits.</td>
</tr>
<tr>
<td>5.3</td>
<td>Dispersion relation and eigenmodes of a transmission line.</td>
</tr>
<tr>
<td>5.4</td>
<td>Eigenmodes of a λ/2 and λ/4 transmission line resonators.</td>
</tr>
<tr>
<td>5.5</td>
<td>Examples of Wigner functions.</td>
</tr>
<tr>
<td>5.6</td>
<td>LC-resonator coupled to semi-infinite transmission line.</td>
</tr>
<tr>
<td>5.7</td>
<td>Beam splitters and circulators.</td>
</tr>
<tr>
<td>5.8</td>
<td>Quadrature measurements with mixers and IQ-mixers.</td>
</tr>
<tr>
<td>6.1</td>
<td>Anharmonic spectrum of a physical qubit.</td>
</tr>
<tr>
<td>6.2</td>
<td>Picture and equivalent circuit of charge and transmon qubits.</td>
</tr>
<tr>
<td>6.3</td>
<td>Energy levels of a charge qubit.</td>
</tr>
<tr>
<td>6.4</td>
<td>Energy levels and eigenfunctions of the transmon qubit.</td>
</tr>
<tr>
<td>6.5</td>
<td>Photography and equivalent circuits of two flux qubits.</td>
</tr>
<tr>
<td>6.6</td>
<td>Energy levels of the rf-SQUID.</td>
</tr>
<tr>
<td>6.7</td>
<td>Inductive energy and eigenstates of the three-junction flux qubit.</td>
</tr>
</tbody>
</table>
### Figures

6.8 Qubit–qubit dipolar interactions. 141
6.9 Qubit coherence times. 149
7.1 Qubits interaction with a transmission line. 158
7.2 Spontaneous emission and scattering. 165
7.3 Spontaneous emission of a photon by a transmon qubit. 168
7.4 Scattering of a propagating photon by a qubit. 171
7.5 Photography of a transmon in a microwave cavity. 173
7.6 Energy levels of the Jaynes–Cummings model. 177
7.7 Transmission spectra for a qubit-cavity setup in the strong and ultrastrong coupling regimes. 179
7.8 Dynamics in the weak and strong coupling regimes. 181
7.9 Spectroscopy schemes for a qubit in a cavity. 185
7.10 Two-tone spectroscopy of a charge qubit. 191
8.1 Quantum computing circuit, schematic notation. 200
8.2 Superconducting qubit quantum register. 203
8.3 Phase gate with two transmons. 210
8.4 Surface code layout and circuits. 226
8.5 Two-qubit gate performance. 231
9.1 Energy gaps, adiabatic evolution, and Landau–Zener processes. 240
9.2 D-Wave architectural details. 254
Tables

<table>
<thead>
<tr>
<th>Table</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1</td>
<td>Critical temperatures of various superconducting materials.</td>
<td>20</td>
</tr>
<tr>
<td>4.1</td>
<td>Equivalence between frequencies and temperatures.</td>
<td>37</td>
</tr>
<tr>
<td>4.2</td>
<td>Mathematical representations of the flux and charge operators.</td>
<td>61</td>
</tr>
<tr>
<td>6.1</td>
<td>Different types of qubits.</td>
<td>108</td>
</tr>
<tr>
<td>7.1</td>
<td>Regimes of the ohmic spin-boson model.</td>
<td>162</td>
</tr>
<tr>
<td>8.1</td>
<td>Most common single-qubit and two-qubit unitaries.</td>
<td>201</td>
</tr>
<tr>
<td>8.2</td>
<td>Completely positive maps for usual error models.</td>
<td>215</td>
</tr>
</tbody>
</table>
Notation

\[ \Delta \hat{O} \quad \text{Uncertainty of operator, } (\Delta \hat{O})^2 = \frac{1}{2} (\hat{O} \hat{O}^\dagger + \hat{O}^\dagger \hat{O}) - |\langle \hat{O} \rangle|^2. \]

\[ \text{Variance of observable, } \hat{O}^\dagger = \hat{O}, \text{ as } (\Delta \hat{O})^2 = (\langle \hat{O} - \langle \hat{O} \rangle \rangle)^2. \]

\[ C \quad \text{Capacitance} \]

\[ E_J \quad \text{Josephson energy} \]

\[ \text{H.c.} \quad \text{Hermitian conjugate, as in } a + \text{H.c.} = a + a^\dagger \]

\[ L \quad \text{Inductance} \]

\[ \mathcal{L} \quad \text{Lagrangian or Lindblad operator} \]

\[ \mathbf{x}, s \ldots \quad \text{Vectors of numbers such as } \mathbf{x} = (x_1, x_2, \ldots, x_N) \]

\[ h \quad \text{Planck constant, } 6.626070040(81) \times 10^{-34} \text{ J} \cdot \text{s} \]

\[ \hbar = \frac{h}{2\pi} \quad \text{Reduced Planck constant} \]

\[ \phi \quad \text{Electric flux on a node or a branch of a circuit (Section 4.3)} \]

\[ \Phi_0 = \frac{\hbar}{2\pi} \quad \text{Magnetic flux quantum, } 2.067833831(13) \times 10^{-15} \text{ Wb} \]

\[ \varphi_0 = \frac{\Phi_0}{2\pi} = \frac{h}{2\pi} \quad \text{Flux-to-phase conversion} \]

\[ \sigma(H) \quad \text{Spectrum or collection of eigenvalues of an operator } H \]

\[ \sigma^x, \sigma^y, \sigma^z \quad \text{Pauli matrices} \]

\[ \mathbf{\sigma} \quad \text{Vector of Pauli matrices } \mathbf{\sigma} = (\sigma^x, \sigma^y, \sigma^z) \]

\[ \text{Linear capacitor} \]

\[ \text{Linear inductor} \]

\[ \text{Nonlinear inductance associated to a Josephson junction} \]

\[ \text{Josephson junction: nonlinear inductor and capacitor in parallel} \]

\[ \text{Constant voltage source} \]