Chapter I
Constructive Ordinals and Π^1_1 Sets

It is shown that a universal quantifier ranging over the real numbers is equivalent in certain circumstances to an existential quantifier ranging over the recursive ordinals, a countable set. Along the way notations for ordinals and the method of defining partial recursive functions by effective transfinite recursion are developed.

1. Analytical Predicates

The analytical predicates are obtained by applying function quantifiers to recursive predicates. Chapter I focuses on analytical predicates in which at most one function quantifier occurs, since in that case an analysis based on ordinals goes smoothly.

1.1 Partial Recursive Functions. Some conventions, occasionally violated, in this book are:

ω is $\{0, 1, 2, \ldots\}$, the set of natural numbers.
b, c, e, m, n are constants that denote natural numbers.
$x, y, z, \ldots$ are variables that range over ω.
$f, g, h, \ldots$ are total functions from ω into ω.
$X, Y, Z, \ldots$ are subsets of ω.
ϕ, ψ, θ are partial functions from ω into ω, that is functions whose graphs are subsets of ω^2.

$\phi(b) \simeq c$ is true iff (if and only if) $\phi(b)$ is defined and equal to c. $\phi(b) \simeq \psi(c)$ iff both $\phi(b)$ and $\psi(c)$ are defined and equal, or neither is defined.

$\{e\}^f$ is the e-th item in the standard enumeration of functions partial recursive in f. There exist a recursive predicate T and a recursive function U, both devised by Kleene, such that

(1) $$\{e\}^f(b) \simeq c \text{ iff } (\mathrm{E}y)[T(\bar{f}(y), e, b, y) \quad \& \quad U(y) = c].$$

$\bar{f}(y)$ encodes $\{\langle i, f(i)\rangle \mid i < y\}$:

$$\bar{f}(y) = \prod_{i<y} p_i^{1+f(i)}.$$

p_i is the i-th smallest prime; $p_0 = 2$. The right side of (1) says there is a computation y derived from the e-th set of equations, and the values of f restricted to $i < y$, whose final outcome is c. All of the above extends to

$$\{e\}^{f_0, \ldots, f_{m-1}}(x_0, \ldots, x_{n-1})$$

for all nonnegative m and n.

1.2 Function Quantifiers. A predicate $R(f, x)$ is recursive if there is an e such that:

(i) $(f)(x)[\{e\}^f(x)$ is defined]; and
(ii) $(f)(x)[R(f, x) \leftrightarrow \{e\}^f(x) = 0]$.

Thus the truth value of $R(f, x)$ is determined by a finite computation. As in subsection 1.1 the definition of recursive predicate extends routinely to predicates of the form $R(f_0, \ldots, f_{m-1}, x_0, \ldots, x_{n-1})$ for all $m, n \geq 0$. For simplicity $R(f, x)$ will be used somewhat ambiguously to denote a recursive predicate with an arbitrary number of function and number variables.

A predicate is *analytical* if it is built up from recursive predicates by application of propositional connectives, number quantifiers and function quantifiers. Thus

(1) $$(\mathrm{Ex})(f)(\mathrm{Eg})R(x, y, f, g, h) \quad \text{and} \quad (\mathrm{Ef})(h)S(f, h, z)$$

are analytical if R and S are recursive. A predicate is *arithmetic* if it is analytical but includes no function quantifiers.

There is an aspect of the classification of predicates which will seem picayune now but which will matter a great deal later. A predicate may be classified by virtue of its explicit form, as were the predicates of (1), or by being proved equivalent to another predicate already classified. For example, the predicate, "f is constructible in the sense of Gödel", is seen to be analytical only after it is shown that every constructible number-theoretic function is constructible via a countable ordinal.

1.3 Theorem (Kleene 1955). *If $P(f, x)$ is analytical, then it can be put in one of the following forms:*

$$A(f,x) \qquad \begin{matrix} (\mathrm{Eg})(y)R(f, x, g, y), & (\mathrm{Eg})(h)(\mathrm{Ey})R(f, x, g, h, y) \ldots \\ (g)(\mathrm{Ey})R(f, x, g, y), & (g)(\mathrm{Eh})(y)R(f, x, g, h, y) \ldots \end{matrix}$$

where A is arithmetic and R is recursive.

Proof. First $P(f, x)$ is put in prenex normal form with a recursive matrix by the usual quantifier manipulations associated with first order logic. Then the resulting prefix is put in one of the desired forms by applying the following rules. K is arbitrary.

(1) $$(x)(\mathrm{Ef})K(f, x) \leftrightarrow (\mathrm{Ef})(x)K((f)_x, x).$$

$(f)_x$ is defined by $(f)_x(y) = f(2^x \cdot 3^y)$.

f is thus interpretable as a code for $\{f_n | n < \omega\}$. Rule (1) is a nontrivial consequence of the axiom of choice. If K is (Eg)(h)$A(f, x, g, h)$ for some arithmetic A, then (1) is provable in Zermelo–Fraenkel set theory (ZF). (See Chapter III, Section 9.) On the other hand there is a K of the form (g)(Eh)$B(f, x, g, h)$ with B arithmetic such that (1) is not provable in ZF.

The dual of (1) is

$$(1^*) \qquad (\mathrm{Ex})(f)K(f, x) \leftrightarrow (f)(\mathrm{Ex})K((f)_x, x).$$

Each of the remaining three rules has a dual.

$$(2) \qquad (\mathrm{Ex})K(x) \leftrightarrow (\mathrm{Ef})K(f(0)).$$

$$(3) \qquad (\mathrm{Ef})(\mathrm{Eg})K(f, g) \leftrightarrow (\mathrm{Ef})K((f)_0, (f)_1).$$

$$(4) \qquad (\mathrm{Ex})(\mathrm{Ey})K(x, y) \leftrightarrow (\mathrm{Ex})K((x)_0, (x)_1).$$

$(x)_i$ is the exponent of p_i, the i-th smallest prime, in the unique factorization of x. Note that the substitution of $(f)_x$ or $(x)_i$ for some of the variables of a recursive predicate leaves it recursive. The following illustrates the normalization of a prefix.

(Ef)(x)(Ey)(h)(z)	is given.
(Ef)(x)(h)(Ey)(z)	by (1*).
(Ef)(g)(h)(Ey)(z)	by (2*).
(Ef)(g)(Ey)(z)	by (3*).
(Ef)(g)(Ey)(h)	by (2*).
(Ef)(g)(h)(Ey)	by (1*).
(Ef)(g)(Ey)	by (3*).

Observe that a prefix can be normalized by deleting all number quantifiers, collapsing each block of function quantifiers of the same sort to a single one of that sort, and adding a single number quantifier on the right dual to the rightmost function quantifier. ☐

Each of the nonarithmetic normal forms of Theorem 1.3 has a Greek name. An analytical predicate in normal form is said to be Σ_n^1 (Π_n^1 respectively) if its prefix begins with an existential (universal respectively) function quantifier and encompasses $n-1$ alternations of function quantifiers. Thus the forms of Theorem 1.3 are

$$\Sigma_1^1, \Sigma_2^1, \Sigma_3^1, \ldots$$

Arithmetic,

$$\Pi_1^1, \Pi_2^1, \Pi_3^1, \ldots$$

A predicate is said to be Δ_n^1 if it is both Σ_n^1 and Π_n^1. The Δ_1^1 predicates will eventually be proved to be the same as the hyperarithmetic predicates.

1.4 Set Quantifiers. The effect of applying function quantifiers to predicates can also be realized by applying set quantifiers. X encodes a function if

(1) $$(x)[E_1 y][2^x \cdot 3^y \in X].$$

If (1) holds, then the function encoded by X is denoted by f_X. Thus

$$f_X(x) = y \leftrightarrow 2^x \cdot 3^y \in X.$$

A predicate $R(X, y)$ is said to be recursive if it is equivalent to some recursive predicate $R(c_X, y)$, where c_X is the characteristic function of X. Let $R_0(X, x, y)$ be a recursive predicate such that (1) is equivalent to $(x)(\mathrm{E}y)R_0(X, x, y)$. The rule for replacing function quantifiers by set quantifiers is:

(2) $$(\mathrm{E}f)K(f) \leftrightarrow (\mathrm{E}X)(x)(\mathrm{E}y)[R_0(X, x, y) \quad \& \quad K(f_X)].$$

Rule (2), and its dual, are all that is needed to transform Theorem 1.3 into Theorem 1.5. It is a fact that the single alternation of number quantifiers occurring in the normal forms of Theorem 1.5 cannot be reduced to a single number quantifier. There exists a recursive predicate $R(X, y, z, x)$ such that $(\mathrm{E}X)(y)(\mathrm{E}z)R(X, y, z, x)$ is not equivalent to $(\mathrm{E}X)(y)S(X, y, x)$ for any recursive S.

A predicate $P(Z, x)$ is *analytical* if it is built up from recursive predicates by application of propositional connectives, number quantifiers and set quantifiers; it is *arithmetic* if no set quantifiers are allowed.

1.5 Theorem (Kleene (1955).) *If $P(Z, x)$ is analytical, then it can be put in one of the following forms*:

$A(Z, x)$	$(\mathrm{E}X)(y)(\mathrm{E}z)R(X, y, z, Z, x)$,	$(\mathrm{E}X)(Y)(\mathrm{E}y)(z)R \ldots$
	$(X)(\mathrm{E}y)(z)R(X, y, z, Z, x)$,	$(X)(\mathrm{E}Y)(y)(\mathrm{E}z)R \ldots$

where A is arithmetic and R is recursive.

There is no harm in mixing set and function variables. Thus a predicate is analytical if it is built up from recursive predicates by application of propositional connectives, number quantifiers, function quantifiers and set quantifiers. It is arithmetic if all quantifiers are number-theoretic. The resulting forms are again denoted by Σ_n^1 or Π_n^1 $(n \geq 1)$.

The most important of all Π_1^1 predicates is: X encodes a countable wellordering. It turns out to be universal Π_1^1, hence not Σ_1^1. It gives rise to a bounding principle with numerous applications. For example, it is used in Chapter IV to compute the Lebesgue measure of a Π_1^1 set of reals.

If $K \subseteq 2^\omega$, then K is said to be Π_n^1 (Σ_n^1 respectively) if $X \in K$ is Π_n^1 (Σ_n^1 respectively). Similar conventions are in force when $K \subseteq \omega$ or $K \subseteq \omega^2$ etc.

1.6 Theorem (Spector 1955). *Suppose* $A(X)$ *is* Σ_1^1.
(i) $\cap\{X|A(X)\}$ *is* Π_1^1.
(ii) If $(E_1X)A(X)$, *then the unique* X *that satisfies* $A(X)$ *is* Δ_1^1.

Proof.
(i) Let B be $\cap\{X|A(X)\}$. Then

$$x \in B \leftrightarrow (X)[A(X) \rightarrow x \in X].$$

(ii) Let C be the unique solution of $A(X)$. Then

$$x \in C \leftrightarrow (\mathrm{E}X)[A(X) \quad \& \quad x \in X]$$
$$\leftrightarrow (X)[A(X) \rightarrow x \in X]. \quad \square$$

In Chapter III, Section 6, it will be shown that every Σ_1^1 set with a non-Δ_1^1 member has a continuum of members. The proof will require more than trivial quantifier manipulations, namely an analysis of Σ_1^1 predicates by means of recursive trees with infinite branching.

Part (i) of Theorem 1.6 is often alluded to as follows: a set (of numbers) is Π_1^1 if it is the closure of a Π_1^1 set under a Σ_1^1 closure condition. A predicate $A(X)$ is a *closure condition* if the intersection of any non-empty collection of solutions of $A(X)$ is a solution of $A(X)$, and if every set (of numbers) is contained in some solution of $A(X)$. Let $A(X)$ be a closure condition. It follows that for each Y there is a least X, call it Y_0, such that $Y \subseteq X$ and $A(X)$:

$$Y_0 = \cap\{X|Y \subseteq X \quad \& \quad A(X)\}.$$

Y_0 is called the closure of Y under A. By Theorem 1.6(i), $Y_0 \in \Pi_1^1$ if $Y \in \Pi_1^1$ and $A(X) \in \Sigma_1^1$, because then

$$Y \subseteq X \quad \& \quad A(X)$$

is Σ_1^1.

1.7 Proposition. $f \in \Sigma_n^1 \leftrightarrow f \in \Pi_n^1 \leftrightarrow f \in \Delta_n^1$.

Proof. Since f is a function,

(1) $$f(x) = y \leftrightarrow (z)[y \neq z \rightarrow f(x) \neq z].$$

If the left side of (1) is Σ_n^1 (Π_n^1 respectively), then the right side is Π_n^1 (Σ_1^n respectively). $\square$

1.8 – 1.12 Exercises

1.8. Show there exists a universal Π_n^1 predicate, that is a Π_n^1 predicate $P(e, f, x)$ such that for each Π_n^1 predicate $Q(f, x)$ there is a c for which $P(c, f, x)$ and $Q(f, x)$ are equivalent for all f and x.

1.9. Show $\Pi_n^1 \subseteq \Sigma_{n+1}^1$, $\Sigma_n^1 \subseteq \Pi_{n+1}^1$, $\Pi_n^1 \nsubseteq \Sigma_n^1$ and $\Sigma_n^1 \nsubseteq \Pi_n^1$.

1.10. Let ω have the discrete topology and ω^ω the product topology. Basic closed subsets of ω^ω can be coded by finite sequences of natural numbers, hence by natural numbers. A closed subset of ω^ω, regarded as an intersection of basic closed sets, can be coded by a subset of ω. Show "X codes a closed subset of ω^ω" is arithmetic. Show "X codes a countable, closed subset of ω^ω" is Π_1^1.

1.11. Let L be a first order language whose set of primitive symbols is recursive. Let $S(X)$ be "X codes a countable set of sentences of L". Show "$S(X)$ and X is consistent (that is yields no contradiction via first order logic)" is arithmetic. Show "$S(X)$ and X has a model" is Σ_1^1.

1.12. Suppose $<_u$ and $<_v$ are order-isomorphic, recursive wellorderings of ω. Show there exists a Δ_1^1 f such that $x <_u y \leftrightarrow f(x) <_v f(y)$ for all $x, y \in \omega$.

2. *Notations for Ordinals*

Suppose $<$ is a wellordering of ω. $<$ is said to be recursive if the predicate $x < y$ is recursive. An ordinal is called recursive if it is finite or the ordertype of a recursive wellordering. The recursive ordinals form a countable, initial segment of the countable ordinals with strong closure properties. They constitute an effective analogue of the countable ordinals. For example, the Cantor–Bendixson analysis of a recursively encodable, countable closed set terminates at a recursive ordinal.

The definition of recursive ordinal is, in a manner of speaking, from above. The question of whether or not a recursive linear ordering ℓ is wellordered is complex. A straightforward resolution would require examination of every function f from ω into ω to see if f defines an infinite descending sequence in ℓ. A more constructive approach would be from below. It seems reasonable to expect that the successor of a constructive ordinal be constructive, and that the limit of a recursive sequence of constructive ordinals be constructive. The constructive approach is made precise with the help of notations. Afterwards it is shown the approaches from above and below yield the same result; the recursive and constructive ordinals coincide. The notion of ordinal notation is useful in proof theory as well as in recursion theory. It facilitates delicate recursions and inductions.

2.1 Kleene's *O*. The formula $x <_O y$ is to be read: x and y are notations for constructive ordinals and x is less than y according to the ordering of notations. The ordering $<_O$ is not linear, because the same ordinal may have two different notations.

The predicate $x <_O y$, regarded as a set of ordered pairs, is the closure of a finite set under a Σ_1^1 closure condition as described in the remarks following subsection 1.6.

The closure condition $A(X)$ has three clauses.

(1) $(u)(v)[\langle u, v\rangle \in X \rightarrow \langle v, 2^v\rangle \in X]$.
(2) $(n)\ [\{e\}(n)$ is defined & $\langle \{e\}(n), \{e\}(n+1)\rangle \in X] \rightarrow (n)[\langle \{e\}(n), 3\cdot 5^e\rangle \in X]$.
(3) $(u)\ (v)\ (w)\ [\langle u, v\rangle, \langle v, w\rangle \in X \rightarrow \langle u, w\rangle \in X]$.

(1) deals with successors, (2) with limits, and (3) with transitivity. All three are positive in nature: if some elements belong to X, then some other elements belong to X. The positiveness of $A(X)$ implies it is a closure condition. Hence there is a least X such that $\langle 1, 2\rangle \in X$ and $A(X)$. $<_O$ is defined to be that least X.

Kleene's O, the set of notations for constructive ordinals, is the field of $<_O$.

A binary relation r is said to be *wellfounded* if there is no f such that $(x)r(f(x+1), f(x))$.

2.2 Theorem

(i) $<_O$ *and* O *are* Π^1_1.
(ii) $<_O$ *is a wellfounded partial ordering*
(iii) If $v \in O$, *then the restriction of* $<_O$ to $\{u|u <_O v\}$ *is linear.*

Proof.

(i) $<_O$ is Π^1_1 by Theorem 1.6(i), since $A(X)$ is arithmetic. Then O is Π^1_1 by rule (1*) of the proof of Theorem 1.3, since $u \in O \leftrightarrow (\mathrm{E}w)[w <_O u \vee u <_O w]$.

(ii) The following *natural enumeration* of $<_O$ is equivalent to a redefinition of $<_O$ by transfinite recursion on the ordinals.

Stage 0: enumerate $1 <_O 2$.

Stage $\delta+1$: enumerate $v <_O 2^v$ and $u <_O 2^v$ if $u <_O v$ was enumerated at stage δ.

Stage λ (limit): enumerate $\{e\}(n) <_O 3 \cdot 5^e$ and $u <_O 3 \cdot 5^e$, if not enumerated earlier, if for each n, $\{e\}(n) <_O \{e\}(n+1)$ has been enumerated earlier, and if for some n, $u <_O \{e\}(n)$ has been enumerated earlier.

By induction on γ, a pair enumerated at stage γ belongs to $<_O$. On the other hand the set of all pairs enumerated is a solution of $A(X)$ and hence contains $<_O$. Also by induction, if $u <_O v$ and $v <_O w$, then $u <_O v$ is enumerated at an earlier stage than $v <_O w$. It follows that $<_O$ is wellfounded, since otherwise there would be an infinite, descending sequence of ordinals.

The natural enumeration also makes clear there is no x such that

(1) $u <_O x \ \& \ x <_O 2^u$, or $x <_O 1$, or
(2) $(n)[\{e\}(n) <_O \{e\}(n+1) <_O x] \ \& \ x < 3 \cdot 5^e$.

Consequently 2^u is said to be the successor of u, and $3 \cdot 5^e$ the limit of $\{e\}(n)\,(n < \omega)$.

(iii) is proved by induction on $<_O$. Assume $u_1, u_2 <_O v$ to check $u_1 <_O u_2$ or $u_1 = u_2$ or $u_2 <_O u_1$. If $v = 2^u$, then (1) implies u_1, $u_2 \leq_O u$ and the desired result follows by induction. If $v = 3 \cdot 5^e$, then apply (2). □

In Part B of this book a generalization of recursive enumerability is offered that allows $<_O$ to be viewed as a higher kind of recursively enumerable relation. The natural enumeration of $<_O$ becomes a proof that $<_O$ is metarecursively enumerable. Elements enter metarecursively enumerable sets by means of metafinite computations, which are infinite but have many of the properties of finite computations.

2.3 Constructive Ordinals. The function $|\ |: O \to$ Ordinals is defined by transfinite recursion on $<_O$.

$$|1| = 0.$$

$$|2^u| = |u| + 1;$$

$$|3 \cdot 5^e| = \lim_{n \to \infty} |\{e\}(n)|.$$

The definition is sound by 2.2(1)–(2). If $u \in O$, then u is said to be a notation for the ordinal $|u|$. An ordinal δ is constructive if $\delta = |u|$ for some $u \in O$. There are no gaps in the constructive ordinals. They form a countable, initial segment of the ordinals. The least non-constructive ordinal is called Church–Kleene omega-one and is written ω_1^{CK}.

The fact that each infinite constructive ordinal has many notations is a consequence of the "approach from below". The choice of a preferred notation for ω is no simple matter. A bad choice for ω might make choices difficult further on (cf. Exercise 2.4). Later it will be seen that there exists a Π_1^1 subset of O, linearly ordered by $<_O$, and of ordertype ω_1^{CK}. Such a subset is called a set of unique notations. It will be defined from above.

2.4–2.6 Exercises

2.4. A path in O is a set $Z \subseteq O$ such that Z is linearly ordered by $<_O$ and $(u)(v)[u <_O v \in Z \to u \in Z]$. A path can be continued if there is a $w \in O$ such that $(u)[u \in Z \to u <_O w]$. Find a path in O of ordertype less than ω_1^{CK} but which cannot be continued.

2.5. Spell out the details omitted from the proof of Theorem 2.2(ii).

2.6. Prove Theorem 2.2(ii) without any reference to ordinals. For example, prove $\sim(\text{Ex})[x <_O 1]$ by showing $A(X) \to A(X - \{\langle x, 1\rangle\})$.

3. *Effective Transfinite Recursion*

Let f map ω into ω, and let $f \restriction n$ denote the restriction of f to $\{m | m < n\}$. To say f is defined by recursion on ω is to say there exists an iterater I such that

$$(1) \qquad f(n) = I(f \restriction n) \text{ for all } n.$$

(1) is called a recursion equation. For each I there is a unique f such that (1) holds. If I is computable, then f is computable by virtue of a straightforward, but limiting, intuition. $f(n)$ is computable from I and the set of previous values, $\{f(m) | m < n\}$. More precisely, $f(n)$ is computed by iterating I n times. The record of that n-fold iteration is the computation of $f(n)$. With this intuition in mind, it appears

farfetched to replace the standard wellordering of ω by some arbitrary wellordering of ω and still expect f to be computable when I is. For one thing, n may have infinitely many predecessors and the iteration of I infinitely many times is not a finite computation. For another, it is no longer clear what is meant by the effectiveness of I, since a typical argument of I in (1) may be an infinite object.

Church and Kleene made the remarkable discovery that (1) remains a valid scheme for defining recursive functions when the standard wellordering of ω is replaced by an arbitrary one, so long as I remains effective in an appropriate sense. It is tempting to think that the wellordering should be recursive, but that limitation is unnecessary, and fortunately so, since many of the applications are to $<_O$. Rogers was the first to use the phrase, "effective transfinite recursion", and to provide a general result similar to Theorem 3.2.

First a technical result of classical recursion theory, Kleene's fixed point theorem. The fact that (1) above has a unique solution f for each I is often described in a set theoretic setting by noting that f is a fixed point of I. The next result is an effective counterpart of the essential existence argument employed in the set theoretic treatment of definition by transfinite recursion.

3.1 Theorem (Kleene). *Suppose $I\colon \omega \to \omega$ is recursive. Then for some c, $\{I(c)\} \simeq \{c\}$.*

Proof. Let t be a recursive function such that

$$\{t(e)\} \simeq \{\{e\}(e)\}$$

for all e. Choose b so that

$$\{b\}(x) = I(t(x)) \tag{1}$$

for all x. Then

$$\{I(t(b))\} \simeq \{\{b\}(b)\} \simeq \{t(b)\}.$$

Thus $t(b)$ will serve for c. □

Note that a fixed point c of I is computable in a uniform manner from a Gödel number of I, since t is independent of I, and since the composition that occurs in (1) is effective. It follows that if I depends effectively on some parameter p, then the fixed point c can be construed as a recursive function of p.

3.2 Theorem. *Let $<_R$ be a wellfounded relation whose field is a subset of ω, and I: $\omega \to \omega$ a recursive function. Suppose for all $e < \omega$ and x in the field of $<_R$, $\{e\}(y)$ defined for all $y <_R x$ implies $\{I(e)\}(x)$ defined. Then for some c, $\{c\}(x)$ is defined for all x in the field of $<_R$, and $\{c\} \simeq \{I(c)\}$.*

Proof. By theorem 3.1 there is a c such that $\{c\} \simeq \{I(c)\}$. Suppose x is a minimal (in the sense of $<_R$) element such that $\{c\}(x)$ is not defined. Thus $\{c\}(y)$ is defined for all $y <_R x$. But then $\{I(c)\}(y) \simeq \{c\}(y)$ is defined. □

Warning: definition by effective transfinite recursion (ETR) is more than an effective version of the set theoretic method of definition by transfinite recursion. There is an element of self-reference in ETR with no counterpart in set theory. There is also a use of indices to transform potentially infinite computations into finite ones. Thus $f(n)$ is computed not by iterating I n times, but by having I act on an index for $f \upharpoonright n$. An excellent example of ETR is the definition of $+_O$.

3.3 Addition of Notations. The key property of $+_O$, the addition function for notations in O, is: if $a, b \in O$, then $a +_O b \in O$ and $|a +_O b| = |a| + |b|$. The definition of $+_O$ by effective transfinite recursion is aimed at realizing this key property.

Let h be a recursive function such that

$$\{h(e, a, d)\}(n) \simeq \{e\}(a, \{d\}(n)) \tag{1}$$

for all e, a, d and n. By the use of pleonasms (that is, each partial recursive function has infinitely many Gödel numbers), it is safe to insist h be one-one from ω^3 into ω.

Let I be a recursive function such that

$$\{I(e)\}(a, b) \simeq \begin{cases} a & \text{if } b = 1 \\ 2^{\{e\}(a,m)} & \text{if } b = 2^m \\ 3 \cdot 5^{h(e,a,d)} & \text{if } b = 3 \cdot 5^d \\ 7 & \text{otherwise.} \end{cases} \tag{2}$$

The first three clauses of (2) mimic the definition of $+$ for ordinals. Thus $\alpha + \beta = \alpha$ if $\beta = 0$, $\alpha + \beta = (\alpha + \gamma) + 1$ if $\beta = \gamma + 1$, and $\alpha + \lambda = \lim_n (\alpha + \gamma_n)$ if $\lambda = \lim_n \gamma_n$. I is recursive, despite the non-recursiveness of $<_O$, because the splitting of O into notations for zero, successors, and limits is effective. Also the instruction coded by $I(e)$ makes sense whether or not a and b belong to O.

By Theorem 3.1, I has a fixed point c. Define $a +_O b$ to be $\{c\}(a, b)$. Since $\{I(c)\} \simeq \{c\}$,

$$a +_O b \simeq \begin{cases} a & \text{if } b = 1 \\ 2^{a +_O m} & \text{if } b = 2^m \\ 3 \cdot 5^{h(c,a,d)} & \text{if } b = 3 \cdot 5^d \\ 7 & \text{otherwise.} \end{cases}$$

Note that $\{h(c, a, d)\}(n) \simeq a +_O \{d\}(n)$ by (1).

Nothing in Theorem 3.1 requires $+_O$ to be defined anywhere. The proof of Theorem 3.2 shows the domain of $+_O$ contains O. (2) has a quirk that compels $+_O$