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PERSPECTIVES IN LOGIC

# *Essential Stability Theory*

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*To my wife, Sally, and our children,*
*Ian, Jessica, Joel and Jeff*

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# Preface

This book grew out of lectures to graduate students in logic at the University of Notre Dame in the academic year 1992-93. The purpose of the course was to bridge the gap between the model theory in a first year graduate logic course (say, the first two chapters in Chang-Keisler) and research papers in stability theory. While the most basic definitions in model theory are repeated in Chapter 1, realistically, I expect the reader to have completed an introductory course in mathematical logic. My intention in writing this book was not to give a comprehensive treatment of elementary stability theory, but to get the student through the basics as quickly as possible. It was also written (hopefully) so that a well-prepared student can begin the book at a chapter appropriate to his or her needs.

Stability theory began in the early '60's with Morley's Categoricity Theorem. (See Section 3.1.) In the period 1965 to 1982 (or so) virtually all attention focused on Shelah's work on (and eventual proof of) Morley's Conjecture. The spectrum function of a complete first-order theory $T$ is a map $I(-,T)$ such that for any cardinal $\lambda$, $I(\lambda,T)$ is the number of models of $T$ of cardinality $\lambda$. In the late '60's Morley conjectured that the spectrum function of a complete countable first-order theory $T$ is nondecreasing on uncountable cardinals; i.e., for all uncountable cardinals $\lambda < \kappa$, $I(\lambda,T) \leq I(\kappa,T)$. Shelah's proof of this conjecture spanned almost 15 years and is the main topic of [She90]. Part of the proof is the development of the forking dependence relation on a stable theory (see Section 5.1). (Examples of this relation are linear dependence in a vector space and algebraic dependence in an algebraically closed field.) The theme through much of Shelah's book is to find subsets of a model of a stable theory on which forking dependence is nice enough to admit a dimension theory (Sections 5.6 and 6.3). (Exactly what is meant by a well-behaved dimension theory is discussed in Section 6.3). Good summaries of this material can be found in [Har93], [She85] and [Hod87].

If I was writing in 1980 the analysis of dependence relations leading up to the proof of Morley's conjecture would be the sole theme of this book. However, in the past 15 years stability theory has grown dramatically in another direction. During the 1980-81 Jerusalem logic year researchers tried to understand Boris Zil'ber's work on the conjecture that a theory categorical in every infinite cardinal is not finitely axiomatizable. It was around this time that Zil'ber completed his proof of the conjecture and Cherlin, Harrington and Lachlan created an independent proof. This work is generally recognized as the birth of geometrical stability theory. (There are earlier results by Zil'ber and others that are properly placed in geometrical stability theory, however, they were not widely known, understood, or seen as closely related at the time.) From here the area took off through further research by the above four mathematicians, Poizat (on stable groups [Poi87]) and myself (see Section 6.2). The entry of Hrushovski around 1984 deepened the area at an tremendous rate. The underlying theme in much of geometrical stability theory is the characterization of certain critical subsets of a model as (essentially) modules or algebraically closed fields. This theme is found throughout Chapter 4 and Section 6.2.

I will not attempt to find a unifying thread in geometrical stability theory and Shelah-style classification theory for two reasons. First, stability theory is growing too rapidly for anyone to come forward and say "this is what it's all about". Secondly, I do not think an all-encompassing theme would be helpful to the reader at the level of this book.

Chapter 1 is simply a quick summary of the prerequisite definitions and theorems. Chapter 2 is a treatment of the classical first-order model theory relevant to stability theory. (Here classical model theory means, with a few exceptions, the model theory that existed prior to Morley's work on his categoricity theorem, circa, 1962.) Morley's Categoricity Theorem is proved in Section 3.1. The student with a good background in classical model theory (Chapters 1 and 2 in [CK73], for example) can begin the book here after absorbing the material on Cantor-Bendixson rank in Section 2.2. The dependence relation induced by Morley rank on a totally transcendental theory is developed in Section 3.3. This theory is applied in the remainder of Chapter 3 to prove the Baldwin-Lachlan Theorem and introduce $\omega$–stable groups.

Geometrical stability theory in an uncountably categorical theory is developed in Chapter 4. This is a good introduction to the area of geometrical stability theory; most of the key concepts are at least mentioned. The deepest results in the chapter are Zil'ber's Ladder Theorems (see Section 4.4) and the group existence results of Section 4.5.

In Chapter 5 we jump to stable theories in general. The forking dependence relation is developed "from scratch" in Section 5.1. The reader with a good understanding of model theory can begin the book in this section, provided she or he has mastered universal domains (Section 3.2) and $T^{eq}$ (Section 4.1). There is very little in the first three sections of this chapter I would term nonessential. In a first reading a student should not feel guilty about skipping the proofs in the sections on prime and saturated models (although the statements of the theorems must be understood). The concepts in Section 5.6 (orthogonality, domination and weight) are at the heart of the dimension theory induced by forking dependence on the universal domain.

The study of superstable theories in Chapter 6 is a natural continuation of the material in Chapter 5. In particular, the dimension theory introduced in Section 5.6 is deepened in the third section on regular types. Geometrical stability theory in the context of a superstable theory of finite rank is introduced in Section 6.2. Section 7.1 contains an application of the dimension theory developed in Chapter 6 to the classification of certain $\omega$−stable theories. Finally, the section on ranks (Section 7.2) contains important facts about Morley rank in an uncountably categorical theory and $\infty$−rank in a unidimensional theory.

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