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PERSPECTIVES IN LOGIC

# *Proper and Improper Forcing*

SAHARON SHELAH
*Hebrew University of Jerusalem*

ASSOCIATION FOR SYMBOLIC LOGIC

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Dedicated to My Beloved Son Omri

מקדש לבני האהוב עמרי

 

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# Introduction

Even though the title of this book is related to the old Lecture Notes "Proper Forcing", it is a new book: not only did it have six new chapters and some sections moved in (and the thirteenth chapter moved out to the author's book on cardinal arithmetic), but the old material and also the new have been revised for clarification and corrected several times.

Now, twenty years after its discovery, I feel that perhaps "proper forcing" is a household concept in set theory, and the reader probably knows the basic facts about forcing and proper forcing. However we demand no prerequisites except some knowledge of naive set theory (including stationary sets, Fodor Lemma, strongly inaccessible, Mahlo, weakly compact etc.; occasionally we mention some large cardinals in their combinatorial definitions (measurable, supercompact), things like $0^\#$ and complementary theorems showing some large cardinals are necessary, but ignorance in those directions will not hamper the reader), and the book aims at giving a complete presentation of the theory of proper and improper forcing from its beginning avoiding the metamathematical considerations; in particular no previous knowledge of forcing is demanded (though the forcing theorem is stated and explained, not proved). This is the main reason for not just publishing the additional material in a shorter book. Another reason is the complaints about shortcomings of the Proper Forcing Lecture Notes.

Forcing was founded by Cohen's proof of the independence of the continuum hypothesis; Solovay and many others developed the theory (works prior to 1977). Particularly relevant to this book are Solovay and Tennenbaum [ST] and Martin and Solovay [MS], Jensen (see [DeJo]), Silver [Si67], Mitchell [Mi1], Baumgartner [B1], Laver [L1], Abraham, Devlin and Shelah [ADS:81].

We do not elaborate here on the history of the subject (but of course there are credits and references to it in each chapter or in its sections, e.g. on Baumgartner's axiom $A$ see VII §4). And in the first two chapters we review classical material.

Our aim is to try to develop a theory of iterated forcing for the continuum. In addition to particular consistency results that are hopefully of some interest, I try to give to the reader methods which he could use for such independence results. Many of the results are presented in an "axiomatic" framework for this reason.

The main aim of the book is thus to enable a researcher interested in an independence result of the appropriate kind, to have much of the work done for him, thus allowing him to quote general results.

We know how for any partial order $P$ (which we call here a forcing notion) we can find an extension $V^P$ of the universe $V$ of set theory (e.g. thinking of $V$ as being countable); this is explained in Chapter I. So for $G \subseteq P$ generic over $V$ (i.e. not disjoint from any dense subset of $P$ from $V$), $V[G]$ is an extension of $V$, a model of set theory with the same ordinals, consisting of sets constructible from $V$ and $G$. By fine tuning $P$ we can get universes of set theory with various desired properties, we are particularly interested in those of the form "for every $x$ there is $y$ such that ...".

So, e.g., for every Souslin tree there is a c.c.c. forcing notion $P$ changing the universe so that it is no longer a Souslin tree, but to prove the consistency of "there is no Souslin tree" we need to repeat it till we "catch our tail". For this an iteration $P * \underset{\sim}{Q}$ is defined, and $(V^P)^{\underset{\sim}{Q}} = V^{P*\underset{\sim}{Q}}$ is proved (i.e. two successive generic extensions can be conceived as one). More delicate is the limit case, and for this case "finite support iteration" $\langle P_i, \underset{\sim}{Q}_j : i \leq \alpha, j < \alpha \rangle$ works; by natural bookkeeping we can consider all Souslin trees and even all Souslin trees in $V^{P_i}$ for $i < \alpha$, and as the c.c.c. is preserved if $\mathrm{cf}(\alpha) > \aleph_1$, since a Souslin tree can be coded as $A \subseteq \omega_1$, we "catch our tail".

In other cases countable support iteration can serve for "catching our tail"; some chain condition is then needed.

The main issue is thus *preservation*, i.e. if each $\underset{\sim}{Q}_i$ has (in $V^{P_i}$) a property, does $P_\alpha$, the limit of an iteration, have it (this depends of course on the kind of the iteration, that is on the support). The most basic property here is not collapsing $\aleph_1$, which is treated for several kinds of iteration, i.e. supports.

Classically, a natural condition guaranteeing that $\aleph_1$ is not collapsed is (the countable chain condition) c.c.c. under finite support iterations; it is preserved (see Chapter II). It is natural to ask that the stationarity of subsets of $\omega_1$ be preserved as well (as if $\langle S_n : n < \omega \rangle$ is a partition of $\omega_1$ to stationary sets, we can force closed unbounded subsets $E_n$ of $\omega_1$ without collapsing $\aleph_1$ (see Baumgartner, Harrington and Kleinberg [BHK]) but in the limit necessarily $\aleph_1$ is collapsed). So it is natural to strengthen this somewhat (to preserve the stationarity of subsets of $\mathcal{S}_{\leq \aleph_0}(\lambda)$ for every $\lambda$), and we get the notion of proper forcing, which is preserved under the countable support iteration (see Chapter III, see alternative proof in XII §1 - and for $\aleph_1$ - free iteration in IX). This is a major notion here and the proof of its preservation serve as paradigm here.

However some forcing notions preserving $\aleph_1$ are not proper: Prikry forcing, Namba forcing. We have to use such forcing (i.e. non-proper) when during the iteration, in some intermediate stage we have to change the cofinality of some uncountable regular cardinal (e.g. $\aleph_2$) to $\aleph_1$. To deal with them we prove the preservation of semiproperness under revised countable support (see Chapter X) and show that Prikry forcing is (always) semiproper and Namba forcing is "often" semiproper. But there may be not semiproper non-proper forcing notions, in particular Namba forcing may be not semiproper (see XII §2), so we introduced properties like the $S$-condition (guaranteeing that no real is added (see Chapter XI)) and UP($\mathbb{I}$) (see Chapter XV), and another one (for continuum $> \aleph_2$) see Chapter XIV.

But of course we really would like to have a general framework for preserving other properties. This is dealt with in Chapter VI §1, §2; a prototypical property is preserving "every new $f \in {}^\omega\omega$ is dominated by an old $g \in {}^\omega\omega$" which is done in V §4 (but this is done only under the assumption that the forcing does not collapse $\aleph_1$ by some of the earlier versions), but includes many examples like "$(f,g)$-bounding for a closed enough family of pairs $(f,g)$". Later we deal with more general theorems (XIII §3, with a forerunner VI §3, which deals e.g. with "no new $f \in {}^\omega\omega$ dominates $F$").

However the case of "no new reals" is somewhat harder and it is treated in several places: in Chapter V §3 (for $S$-complete forcing, $S \subseteq \mathcal{S}_{\leq\aleph_0}(\lambda)$ stationary), in Chapter V §6, §7 (for $(<\omega_1)$-proper $\mathbb{D}$-complete (e.g. $\mathbb{D}$ is a simple $\aleph_1$-completeness system)), Chapter VIII §4 (for a generalization, in particular $\mathbb{D}$ is a simple 2-completeness system) and Chapter XVIII §2 (for essentially strongly proper forcing notions). Some other properties do not fall (or are not presented) under any of those cases: strongly proper (see IX, VI §6, IX §4), not collapsing $\aleph_2$ for $\aleph_1$-complete forcing when we use mixed support (VIII §1).

In all iterations somewhere we need to prove a suitable chain condition in order to catch our tails; for finite support iteration this is done directly (II), for countable support iteration see III §4, and more VII §1, VIII §2, and for revised countable support later in XVII §4 this plays a central role.

Every iteration theorem + chain condition gives the consistency of an axiom, see VII §3, VIII §4, XVII §1, §2, §3. This stresses the problem of having a general iteration theorem for "continuum $\geq \aleph_3$".

Of course much of the book deals with specific problems which serve both as an illustration of the methods and for their interest per se (see the annotated content and the separate introductions to the chapters).

The mathematical work was done between 1977 and 1989; the author approaches other aspects of our subject in Judah and Shelah [JdSh:292], [Sh:630] (on nicely defined forcing, e.g. Borel), Rosłanowski and Shelah [RoSh:470] (on even more nicely defined forcing, quite explicitly in fact), [Sh:176] §7, §8, Goldstern and Shelah [GoSh:295] (on amalgamation and projective sets in the final model) and [Sh:311] (continuing Chapter XV), [Sh:587], [Sh:F259], [Sh:655] (on replacing $\aleph_0, \aleph_1$ by $\lambda, \lambda^+$) and [Sh:592] (more on FS iteration of c.c.c. forcing notions).

Prerequisites: we assume that the reader has some knowledge of naive set theory (including stationary sets, Fodor lemma, etc.). The metamathematical side is avoided by stating the forcing theorem without proof in Chapter I. If you have read Jech [J] or Kunen [Ku83] you should be well prepared. Several places present some preservation theorems with less generality and they may be of some help to the reader (and they can be treated as good preparation for this book too). Let us list some of them: Baumgartner [B3], Abraham [Ab], Bartoszyński and Judah [BaJu95], Goldstern [Go], Jech [J86], Judah and Repicky [JuRe].

* * *

There are many people who gave indispensable help for the book in various stages.

Concerning the old “Proper Forcing” I heard from Baumgartner the idea of avoiding the metamathematical side, Azriel Levy, who has a much better name than the author in such matters, made notes from the lectures in the Hebrew University, rewrote them, and they appear as Chapters I, II, and part of III. These chapters were somewhat corrected and expanded by Rami Grossberg and the author. Most of XI §1–5 were lectured on and (the first version) written up by Shai Ben David. And most of all Rami Grossberg has taken care of it in all respects and Danit Sharon typed it.

Concerning the present work, Azriel Levy has helped in transforming the book from Troff to TEX. Menachem Kojman in addition wrote up first version of section I §7 from my lectures in the eighties.

Various parts of the book benefitted a lot from proofreading and pointing gaps by my students, postdocs and co-workers. All of them contributed in various ways and I am very grateful for their help, though I failed to make a complete list of the contributors.

Most of all Martin Goldstern has helped in some very distinct capacities. Not only did he proofread and revise several chapters but also with his magic

touch in various TEX-nical aspects he made the appearance of this book possible.

Last but not least I thank Hagit Levy and Gosia Rosłanowska for typing and retyping and retyping ... the book.

Thank you all.

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† This footnote was omitted by mistake in [Sh:b].

# Notation

Natural numbers are denoted by $k, \ell, m, n$.

Ordinals are denoted by $i, j, \alpha, \beta, \gamma, \delta, \xi, \zeta, \theta$ where $\delta$ is reserved for limit ordinals.

Cardinals (usually infinite) are denoted by $\lambda, \mu, \kappa, \chi$. Let $\aleph_\alpha$ be the $\alpha$'th infinite cardinal, $\omega_\alpha = \aleph_\alpha$, $\omega = \omega_0$. Let ${}^\beta\alpha = \{f : f$ a function from $\beta$ to $\alpha\}$, ${}^{\beta>}\alpha = \bigcup_{\gamma<\beta} {}^\gamma\alpha$. For sequences of ordinals (i.e., members of some ${}^\beta\alpha$), $\ell g(\eta) = \mathrm{Dom}\eta$, $\eta(i)$ the $i$'th ordinal so $\eta = \langle \eta(i) : i < \ell g(\eta)\rangle$. We write also $\langle \eta(0), \dots, \eta(n)\rangle$ or $\eta(0), \dots, \eta(n)$ as seems fit. We denote sequences, usually of ordinals, by $\eta, \nu, \rho$ and also $\tau$. The concatanation of $\eta$, $\nu$ is denoted by $\eta\hat{\ }\nu$ Let c.l.u.b. or club mean closed unbounded.

Let $|A|$ denote the cardinality of the set $A$, $\mathcal{P}(A)$ denote the power set of $A$, and $\mathrm{cf}(\alpha)$ the cofinality of $\alpha$.

"A real" means here a subset of $\omega$, or its characteristic functions. The quantifiers "$\exists^* n$", "$\forall^* n$" (sometimes written $\exists^\infty$ and $\forall^\infty$, respectively) are abbreviations for "for infinitely many $n \in \omega$" and "for all but finitely many $n \in \omega$", respectively.

Let $\varphi, \psi$ denote first order formulas. Let $\varphi(x_0, \dots, x_{n-1})$ means every free variable of $\varphi$ appears in $\{x_0, \dots, x_{n-1}\}$.

Let $P$ (and also $Q$ and $R$) denote a partially ordered set or even a quasi ordered set (i.e., $p \leq q \leq p$ does not necessarily imply $p = q$). We call such $P$ a forcing notion, and assume $\emptyset$ ot $\emptyset_P$is a minimal member of $P$. We use $G$ for a generic subset of $P$, (usually $G_\alpha$ or $G_{P_\alpha}$ for a generic subset of $P_\alpha$), (for a definition of generic see I §1). Let $p, q, r$ denote members of forcing notions, we say $p, q$ are incompatible in $P$ if they have no common upper bound in $P$.

We do not distinguish strictly between a model $M$, or a forcing notion $P$, and their universe. $\bigtimes_{i\in I} A_i$ is the cartesian product sometimes also denoted by $\prod_{i\in I} A_i$. Distinguish (in principle) it from $\prod_{i\in I} \mu(i)$ (multiplication of cardinals). We shall not distinguish notationally between multiplication of cardinals or ordinals. For an uncountable cardinal $\lambda$ of uncountable cofinality, $\mathcal{D}_\lambda$ stands

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for the filter generated by the closed unbounded subsets of $\lambda$; when $S \subseteq \lambda$ is stationary (also denoted by $S \not\equiv 0 \bmod \mathcal{D}_\lambda$) $\mathcal{D}_\lambda + S$ is the filter generated by $\mathcal{D}_\lambda \cup \{S\}$.

For ordinals $\alpha, \beta$ such that $\beta < \alpha$ we let $S^\alpha_\beta = \{\gamma < \aleph_\alpha : \mathrm{cf}\gamma = \aleph_\beta\}$ but when $\mu, \lambda$ are (infinite) cardinals such that $\mu < \lambda$ then $S^\lambda_\mu = \{\gamma < \lambda : \mathrm{cf}\lambda = \mu\}$ when no confusion arise.

We made a special effort to uniformize the notation we use but still there may be some exceptions in the chapters, for example in Chapter III, $\mathcal{S}_{\aleph_0}(A)$ is used to denote the family of countable subsets of $A$ (see Definition III 1.2) but the same family in Chapter V is denoted by $\mathcal{S}_{<\aleph_1}(A)$ (see Definition V 3.3). So since some of the notions are redefined the reader is advised to check the nearest definition rather than the first.

Models are denoted by letters $M$, $N$ perhaps with an index. We shall not always distinguish between the model and its universe, but always $|M|$ will denote the universe of $M$ and $\|M\|$ its cardinality.

# Content by Subject

## 1. Results Outside Set Theory

In the Appendix, Sect. 2 there are results on the power of Ext$(G, \mathbb{Z})$ assuming various weak diamonds.

## 2. Results in Naive Set Theory

In the Appendix §1 we investigate weak variants of the diamond (continuing Devlin and Shelah [DvSh:65].)

In the Appendix §3 we prove that CH implies some kind of weak diamond to $S^2_1$ (and generalizations).

In IX §3 we discuss various specializations of Aronszajn trees.

In XIII §4 a sufficient condition for the existence of Ulam filters is given.

## 3. Basics of Forcing

In Chapter I we explain how to use forcing and discuss some basic forcing. In Chapter II we explain iteration with $< \kappa$-support, (in particular finite support) deal for example with Martin Axiom, and more examples.

## 4. Specific Independence Results on Trees

In III 5.4 we present a proof that every $\aleph_1$-Aronszajn tree can be specialized by a c.c.c. forcing notion (and generally in III §5 deal with $\kappa$-trees).

In III §6 we present a proof of the consistency of " ZFC+$2^{\aleph_0} = \aleph_2$+ there is no $\aleph_2$-Aronszajn trees".

In V §6, §7, we present a proof of CON (ZFC+G.C.H.+ every Aronszajn tree is special) which seems to us more adaptable to further needs (e.g. $2^{\aleph_1}$ large).

In V §8 we present a proof of the consistency of the Kurepa hypothesis.

In VII §3 we prove consistency results on Aronszajn trees strengthening the previous ones, motivated by general topology. This implies the consistency of G.C.H. + there is a countably paracompact regular space which is not normal (see VII 3.25).

In VII §3 we present a proof of the consistency of a strengthening of CON (ZFC+G.C.H. + there is no $\aleph_1$-Kurepa tree.)

In VIII §3 we prove the consistency of ZFC+CH+SH+$2^{\aleph_1} > \aleph_2$.

In IX§4 we prove that SH$\not\Rightarrow$ " every Aronszajn tree is special" and variations.

## 5. Theorems on $\aleph_1$-Complete Forcing

In VIII §1 we prove that we can iterate $\aleph_2$-complee forcing and $\aleph_1$-complete forcing satisfying a strong $\aleph_2$-chain condition, without collapsing $\aleph_1$ and $\aleph_2$.

In VIII 2.7A we remark on another strong $\aleph_2$-chain condition preserved by CS iteration.

## 6. Chain Conditions

The c.c.c. and $\kappa$-c.c. are presented in Chapter II and the preservation of the c.c.c. by FS is proved there.

In III 4.1 we prove that a CS iteration of proper forcing notions of power $< \kappa$, $\kappa$ regular $(\forall \mu < \kappa)\ \mu^{\aleph_0} < \kappa$, of length $\kappa$ satisfies the $\kappa$-c.c. (by proving that if the length is $< \kappa$, it has a dense subset $< \kappa$).

Of course IV is dedicated to oracle-c.c.

Lemma V 1.5 proves the $\aleph_2$-c.c. of a CS iteration of $E$-complete forcing notions each of power $\aleph_1$, (assuming CH).

VII §1 deals with a strong $\kappa$-chain (e.c.c.), such that if we have a CS iteration of length $\kappa$ with the condition we use for not adding reals (in V§7, VIII§4) then the forcing satisfies the $\kappa$-c.c. So this helps to get consistency results with ZFC+CH.

In VII §2 we deal with $\kappa$-pic (= $\kappa$-properness isomorphism conditions). If $P$ satisfies it, then $P$ satisfies the $\kappa$-c.c., and adds $< \kappa$ reals, and an iteration of length $\kappa$ still satisfies the $\kappa$-c.c. This helps to get consistency results with ZFC+$2^{\aleph_0} = \aleph_2 + 2^{\aleph_1} = \lambda$ (so we start with $V \models$ "$CH + 2^{\aleph_1} = \kappa$" and this holds for the intermediate stages). Application is starting with $V \models$ "CH+$2^{\aleph_1} = \kappa$" and use a CS iteration $\bar{Q} = \langle P_i, Q_i : i < \omega_2 \rangle$ to specialize all Aronszajn trees without adding reals.

On the $\kappa$-c.c. for RCS iteration of length $\kappa$ see X §5 (5.3, 5.4) and XI 6.3(2).

On $\lambda$-c.c. for $\kappa$-RS iterations see XIV.

**7. Preservation of Properness and Variants**

As the book was written in the generic way, i.e. as the author advance, there are several such proofs. In III §3 the preservation of properness by CS iteration is proved. In IX §2 the preservation of properness under $\aleph_1$-free limit is proved. In X §2 the preservation of properness and semi-properness under RCS is proved. (remember that semi-proper forcing may change the cofinality of some regular $\lambda > \aleph_1$ to $\omega$). In XII §2 the preservation of properness under CS iteration is reproved, using the definition of properness by games (similarly for semiproperness) .

We deal with preservation of $\alpha$-properness and $(\omega, 1)$- properness in V §2, §3, X §7.

**8. Consistency Results on the Uniformization Properties and Variants**

In II §4 we prove the consistency of "some family $\mathcal{P}$ of $\aleph_1$ subsets of $\omega$ has $k$-uniformization property" i.e. if $f_A : A \to k$ for $A \in \mathcal{P}$ then for some $f : \omega \to k$ we have

$$A \in \mathcal{P} \Rightarrow (\forall^* n \in A)(f(n) = f_A(n)).$$

In V§1, we prove the consistency with G.C.H. of "for some stationary $S \subseteq \omega_1$, $\langle A_\delta : \delta < \omega_1 \rangle$ have the $\aleph_0$-uniformization property if each $A_\delta$ is a set of order type $\omega$, $\sup(A_\delta) = \delta$".

In VIII §4 we prove the consistency with G.C.H. of $\neg\Phi^3_{\aleph_2}$. Moreover, e.g.: for $\langle A_\delta : \delta < \omega_1 \text{ limit}\rangle$ as above, $n_\delta < 3$ then for some $f : \omega_1 \to 3$, for every $\delta < \omega_1$ limit, for some $m_\delta \in \{0, 1, 2\} \setminus \{n_\delta\}$, we have $(\forall^* \alpha \in A_\delta)(f(\alpha 0 = m_\delta)$.

**9. Consistency of "Large Ideals"**

In XIII we get consistency results on $\omega_1$, in XVI use smaller ones, in XIV with larger continuum.

In XVII §4 we deal with properties consistent with $\neg 0^{\#}$ (on $\omega_1$ being an $\alpha$-th function from $\omega_1$ to $\omega_1$ even $\alpha = 2^{\aleph_1}$).

## 10. Other Consistency Results

In III §7 we deal with " for a family of $\aleph_1$ countable subsets $A_i$ of $\omega_1 (i < \omega_1)$, order type $(A_i) < \mathrm{Sup} A_i$, there is a club $C \subseteq \omega_1$ $\bigwedge_i \mathrm{Sup}(C \cap A_i) < \mathrm{Sup} A_i$.

In VI §4 we prove the consistency of " there is no $P$-point".

In VI §5 we prove the consistency of " there is a unique Ramsey ultrafilter (on $\omega$)".

In X, XI we prove various independence results on $\omega_2$, read XI §1, X 8.4, and XI §7's theorem.

In VII §3, §4, many applications are listed.

In XVIII §4 we prove the consistency of "there is a unique $P$-point (on $\omega$)" (which necessarily is a Ramsey ultrafilter).

## 11. Other Preservations of Generalizations of Properness

In V §3 we prove the preservation of $\omega$-properness+ the $\omega^\omega$-bounding property (properness suffices - see VI).

In VI §1, §2, §3, XVIII §3 we deal with general preservation theorems, e.g. of covering models, hence of various specific properties which can be formulated this way ($F$-bounding properties, Sacks property, Laver property, $PP$-property, $D$ generates a Ramsey ultrafilter, $D$ generates a $P$-point (ultrafilter)). Now VI §2 relies on VI §1 to give specific results (using covering models); a different approach, guaranteeing only preservation continue to hold in limit stages, is VI §3, which apply e.g. to "$F \subseteq {}^\omega\omega$ is not dominated by $\cdot$"; this is further developed in XVIII §3.

In IX §4 we deal with preservation of "$(T^*, S)$-preserving" which means $T^*$ looks like Souslin trees at levels $\delta \notin S$, and in the end we comment on possible generalizations.

Not adding reals is dealt with in V §1, §2 and mainly V §7, VIII §4 and X §7 and in XVIII §2.

In V §1 and X §3 we deal with preservation of generalizations of $\aleph_1$-completeness.

In XI §5, §6 we deal with the preservation of the $S$-condition (always satisfied by Nm which change the cofinality of $\aleph_2$ to $\aleph_0$ but guarantees reals are not added).

In XV §3 we deal with a generalization, i.e. proving preserving of a condition implying $\aleph_1$ not collapsed, which is satisfied by all proper forcing notions and all forcing satisfying the $S$-condition.

## 12. Forcing Axioms

On consistency, see III §4, VII §2, VIII §3, X 2.6, XIII, XIV, XVI §2. Also XVII §3 (SPFA$\nvdash$ weak Chang conjecture), XVII §1 (SPFA$\equiv$ MM), XVII §2 (SPFA$\nvdash$ PFA$^+$)

## 13. Counterexamples

In VII §5 we build an iteration $\langle P_n, \underset{\sim}{Q}_n : n < \omega \rangle$ such that each $\underset{\sim}{Q}_n$ does not collapse any stationary subset of $\omega_1$; but any limit we take collapses $\aleph_1$.

In III §4 we build (in ZFC) a forcing notion of power $\aleph_1$, not collapsing $\aleph_1$ but also not preserving the stationarity of some $S \subseteq \omega_1$.

Examples of iteration $\langle P_n, \underset{\sim}{Q}_n : n < \omega \rangle$ each $\underset{\sim}{Q}_n$ is $\alpha$-proper for each countable $\alpha$ not adding reals, but any limit we take that collapses $\aleph_1$ is presented in V 5.1 using Appendix §1 (really the previous result of Devlin and Shelah [DvSh:65]).

In XII §2 we show that many forcing notions satisfy the conditions from XI but are not semiproper. In fact if there is an $\{\aleph_1\}$-semiproper forcing notion changing cofinaltity of $\aleph_2$ to $\aleph_0$ (e.g. if Nm is $\{\aleph_1\}$-semiproper then Chang conjecture holds.

Examples of iteration $\langle P_n, \underset{\sim}{Q}_n : n < \omega \rangle$, each $\underset{\sim}{Q}_n$ is a $\alpha$-proper and is a $\mathbb{D}$-complete for some simple $\aleph_1$-complete completeness system but any limit adds reals is presented in XVIII §1.

# Annotated Content

## I. FORCING, BASIC FACTS

### §0. Introduction

### §1. Introducing Forcing

We define generic sets, names for a forcing notion, and formulate the forcing theorems.

### §2. The Consistency of CH

Our aim is to construct by forcing a model of ZFC were CH holds. First we explain the problem of not collapsing cardinals, and second prove that $\aleph_1$-complete forcing notion does not add reals.

### §3. On the Consistency of the Failure of CH

We construct a model of ZFC in which the Continuum Hypothesis fails; define the c.c.c., prove that forcing with c.c.c. forcing preserves cardinalities and cofinalities, and prove also the $\Delta$-system lemma for finite sets.

### §4. More on the Cardinality $2^{\aleph_0}$ and Cohen Reals

We construct for every cardinal $\lambda$ in $V$ which satisfies $\lambda^{\aleph_0} = \lambda$ a model $V[G]$ such that $V[G] \models 2^{\aleph_0} = \lambda$. Also Cohen reals are defined.

### §5. Equivalence of Forcing Notions, and Canonical Names

We define when two forcing notions are equivalent, introduce canonical names and prove that for every $P$-name $\underset{\sim}{\tau}$ there is a canonical $P$-name $\underset{\sim}{\sigma}$ such that $\Vdash_P$ "$\underset{\sim}{\tau} = \underset{\sim}{\sigma}$".

### §6. Random Reals, Collapsing Cardinals and Diamonds

We introduce random reals and the Levy collapse, and prove that for regular $\lambda$ $\mathrm{Levy}(\aleph_0, < \lambda)$ satisfies the $\lambda$-c.c. For every uncountable regular $\lambda$ and a stationary $S \subseteq \lambda$ define a forcing notion $P$ which preserve the regularity of $\lambda$ and stationarity of $S$ add no bounded subsets to $\lambda$ and such that $V^P \models \diamondsuit_S$.

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More Information



### §7. ♣ Does Not Imply Diamond

♣ is a weak relative of diamond, for $S \subseteq \lambda$ stationary $♣(S)$ says that we can find $\langle A_\delta : \delta \in S \rangle$ $A_\delta \subseteq \delta = \sup(A_\delta)$, and for every unbounded $A \subseteq \lambda$ for some ($\equiv$ stationarily many) $\delta \in S$, $A_\delta \subseteq A$. If $CH$, $♣(\aleph_1) \equiv \diamondsuit_{\aleph_1}$, so we prove the consistency of $♣(\aleph_1) + \neg CH$, but forcing three times.

Start with $V \vDash GCH$, for $\diamondsuit_s$ for $S = \{\delta < \aleph_2 : \mathrm{cf}(\delta) = \aleph_0\}$, using it construct a "very good" $♣(S)$ sequence $\langle A_\delta : \delta \in S \rangle$. Then force by adding $> \aleph_2$ subsets of $\aleph_1$ by countable conditions, $\bar{A}$ was constructed to withstood it, lastly collapse $\aleph_1$ to $\aleph_0$ by $\mathrm{Levy}(\aleph_0, \aleph_1)$. As any new unbounded subset of $\aleph_2$ contains an old one, $\bar{A}$ still witness $♣(S)$ but now $S$ is a stationary subset of $\aleph_1$ of the last universe.

## II. ITERATION OF FORCING

### §0. Introduction

### §1. The Composition of Two Forcing Notions

Composition of two notions and state the associativity lemma are defined.

### §2. Iterated Forcing

We define iterated forcing, and prove that the c.c.c is preserved by FS (finite support) iteration.

### §3. Martin's Axiom and Few Applications

We prove that ZFC $+2^{\aleph_0} > \aleph_1$+MA is consistent. Use MA to prove many simple uniformization properties.

### §4. The Uniformization Property

Here we deal with more general uniformization properties some of which contradict MA. We strenghten the demand of almost disjointness to being a kind of tree and prove the consistency of a version contradicting MA.

### §5. Maximal Almost Disjoint Families of Subsets of $\omega$

A maximal almost disjoint (MAD) subset of $\mathcal{P}(\omega)$ is a family of infinite subsets of $\omega$ such that the intersection of any two members is finite and maximal with this property. We prove using MA that every MAD set has cardinality $2^{\aleph_0}$. Also the other direction: for every $\aleph_1 \leq \lambda < 2^{\aleph_0}$ there exists a generic extension of $V$ by c.c.c. forcing such that in it there exist mad set of power $\lambda$.

## III. PROPER FORCING

### §0. Introduction

### §1. Introducing Properness

We define "$P$ is a proper forcing notion ", prove some definitions are equivalent (and deal with the closed unbounded filter $\mathcal{D}_{\aleph_0}(\lambda)$).

### §2. More on Properness

We define "$p$ is $(N, P)$-generic " and deal more with equivalent definitions of properness.

### §3. Preservation of Properness Under CS Iteration

We prove the theorem mentioned in the title, CS is countable support.

### §4. Martin's Axiom Revisited

We discuss the popularity of the c.c.c., whether we can replace it by a more natural and weaker condition. We give a sufficient condition for a countable support iteration of length $\kappa$ to satisfy the $\kappa$-c.c. We prove the consistency (assuming existence of an inaccessible cardinal) of "ZFC + $2^{\aleph_0} = \aleph_1 +$ Ax[ for forcing notions not destroying stationary subsets of $\omega_1$ of cardinality $\aleph_1$]". We show that the last demand cannot be replaced by "not collapsing cardinalities or cofinalities".

### §5. On Aronszajn Trees

We define $\kappa$-Aronszajn and $\kappa$-Souslin trees. We then present existence theorems (for $\lambda^+$ when $\lambda = \lambda^{<\lambda}$) and prove that under MA every Aronszajn tree is special.

### §6. Maybe There Is No $\aleph_2$-Aronszajn Tree

We prove the consistency of ZFC $+2^{\aleph_0} = \aleph_2 +$ there is no $\aleph_2$-Aronszajn tree, the method being collapsing successively all $\lambda, \aleph_1 < \lambda < \kappa$ ($\kappa$ a weakly compact cardinal) and treating every potential initial segment of an $\aleph_2$-Aronszajn tree to ensure it will not actually become an initial segment of such a tree.

### §7. Closed Unbounded Subsets of $\omega_1$ Can Run Away from Many Sets

We prove the consistency of ZFC $+2^{\aleph_0} = \aleph_2$ with if for $i < \omega_1$, $A_i \subseteq \omega_1$ has order type $< \mathrm{Sup} A_i$, then for some closed unbounded $C \subseteq \omega_1, (\forall i)[\mathrm{Sup}(C \cap A_i) < \mathrm{Sup} A_i]$.

## IV. ON ORACLE-C.C., THE LIFTING PROBLEM OF THE MEASURE ALGEBRA, AND "$\mathcal{P}(\omega)$/FINITE HAS NO NON-TRIVIAL AUTOMORPHISM"

### §0. Introduction

The oracle-c.c. method enables us to start with $V \models \diamondsuit_{\aleph_1}$ and extend the set of reals $\omega_2$-times (by iterated forcing), in the intermediate stages $\diamondsuit_{\aleph_1}$ holds, and we omit types of power $\aleph_1$ along the way, i.e. promise that some intersections of $\aleph_1$ Borel sets remain empty.

### §1. On Oracle Chain Condition

One way to build forcing notions satisfying the $\aleph_1$-c.c., is by successive countable approximations including promises to maintain the predensity of countably many subset, many times using the diamond. We formalize a corresponding property ($\bar{M}$-c.c., $\bar{M}$ an oracle) and prove the equivalence of some variants of the definition.

### §2. The Omitting Type Theorem

We prove that if the intersection of $\aleph_1$ Borel sets is empty and even if we add a Cohen real it remains empty, (and $\diamondsuit_{\aleph_1}$) *then* for some oracle $\bar{M}$, for every forcing notion $P$ satisfying the $\bar{M}$-c.c., in $V^P$ the intersection of the Borel sets (reinterpreted) is still empty.

### §3. Iterations of $\bar{M}$-c.c. Forcings

We show that for Finite Support iteration $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i : i < \alpha \leq \omega_2 \rangle$, if $\bar{M}_i \in V^{P_i}$ is an $\aleph_1$-oracle large enough for $\langle \langle \bar{M}_j, P_j, \bar{Q}_j \rangle : j < i \rangle$, and $\underset{\sim}{Q}_i$ satisfies the $\bar{M}_i$-c.c. *then* $P_\alpha = \text{Lim}\bar{Q}$ satisfies the $\bar{M}_0$-c.c. The first three sections give the exact formulation of the aim stated in the introduction and prove that it works.

**§4. The Lifting Problem of the Measure Algebra**

We show how to apply the method described in §1 - §3 in order to get a model in which the natural homomorphism $\mathcal{B} \to \mathcal{B}/$ (measure zero sets) does not lift. Where $\mathcal{B}$ is the Boolean algebra of Borel sets of reals.

**§5. Automorphisms of $\mathcal{P}(\omega)$/finite**

We use our method to prove that the Boolean algebra "power set of $\omega$ divided by the ideal of finite sets" can have only trivial automorphisms, where those are defined as the ones induced by permutations of $\omega$ or "almost" permutation of $\omega$. This is equivalent to having the topological space $\beta(\mathbb{N}) \setminus \mathbb{N}$ having only trivial autohomeomorphisms. However a main lemma is delayed to the next section.

**§6. Proof of Main Lemma 5.6**

The point missing in §5 is: if $F$ is an automorphism of $\mathcal{P}(\omega)$/finite, $\bar{M}$ an $\aleph_1$-oracle, then there is forcing notion $P$ satisfying the $\bar{M}$-c.c., and a $P$-name $\underset{\sim}{X}$ of a real such that in $V^P$, for no $Y \subseteq \omega$, $(\forall A, B \in \mathcal{P}(\omega)^V)[X \cap A =_{ae} B \Rightarrow \underset{\sim}{Y} \cap F(A) =_{ae} F(B)]$, moreover even a Cohen forcing does not introduce such a $Y$, where $=_{ae}$ means equal modulo finite. We try to build such $P$, $\underset{\sim}{X}$ and prove that if we always fail $F$ is trivial.

## V. $\alpha$-PROPERNESS AND NOT ADDING REALS

**§0. Introduction**

**§1. $\mathcal{E}$-Completeness – a Sufficient Condition for Not Adding Reals**

We define what it means to be $\mathcal{E}$-complete e.g., if $P \subseteq ({}^{\omega_1>}2, \triangleleft)$, $\mathcal{E} \subseteq \omega_1$ stationary and $f_n \subseteq f_{n+1} \in P$, $\text{Sup}(\text{Dom} f_n) \in \mathcal{E}$. We show that properness + $\mathcal{E}$-completeness are preserved by CS iteration and get corresponding Axiom. We also introduce a forcing axiom which is consistent with CH use it to prove