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978-1-107-16836-7 — Proper and Improper Forcing
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I. Forcing, Basic Facts

§0. Introduction

In this chapter we start by introducing forcing and state the most important theorems on it (done in §1); we do not prove them as we want to put the stress on applying them. Then we give two basic proofs:
in §2, we show why CH (the continuum hypothesis) is consistent with ZFC, and in §3 why it is independent of ZFC. For this the $\aleph_1$-completeness and c.c.c.(=countable chain conditions) are used, both implying the forcing does not collapse $\aleph_1$ the later implying the forcing collapse no cardinal. In §4 we compute exactly $2^{\aleph_0}$ in the forcing from §3 (in §3 we prove just $V[G] \models$ "$2^{\aleph_0} \geq \lambda$"; we also explain what is a "Cohen real"). In §5 we explain canonical names.

Lastly in §6 we give more basic examples of forcing: random reals, forcing diamonds. The content of this chapter is classical, see on history e.g. [J]. (Except §7, 7.3 is A. Ostaszewski [Os] and 7.4 is from [Sh:98, §5], note that later Baumgartner has found a proof without collapsing and further works are:

P. Komjáth [Ko1], continuing the proof in [Sh:98] proved it consistent to have MA for countable partial orderings $+\neg$CH, and $\clubsuit$. Then S. Fuchino, S. Shelah and L. Soukup [FShS:544] proved the same, without collapsing $\aleph_1$ and M.Džamonja and S.Shelah [DjSh:604] prove that $\clubsuit$ is consistent with SH (no Souslin tree, hence $\neg$CH).)

§1. Introducing Forcing

1.1 Discussion. Our basic assumption is that the set theory ZFC is consistent. By Godel's completeness theorem it has a countable model. We make the following further assumptions about this model.

(a) The membership relation of the model is the real membership relation; and therefore the model is of the form $(V, \in)$.
(b) The universe V of the model is a transitive set, i.e., $x \in y \in V \to x \in V$.

Assumptions (a) and (b) are not essential but it is customary to assume them, and they simplify the presentation. So "V a model of ZFC", will mean "a countable model of ZFC satisfying (a) and (b)", and the letter V is used exclusively for such models.

Cohen's forcing method is a method of extending V to another model $V^\dagger$ of ZFC. It is obvious that whatever holds in the model $V^\dagger$ cannot be refuted by a proof from the axioms of ZFC, and therefore it is compatible with ZFC. If we show that a statement and its negation are both compatible with ZFC then we know that the statement is undecidable in ZFC.

Why do we look at extensions of V and not at submodels of V? After all, looking at subsets is easier since their members are already at hand: To answer this question we have to mention Godel's constructibility. The constructible sets are the sets which must be in a universe of set theory once the ordinals of that universe are there. Godel showed that the class L of the constructible sets is a model of ZFC and that one cannot prove in ZFC that there are any sets which are not constructible. Therefore, for all we know, V may contain only sets which are constructible and in this case every transitive subclass $V^\dagger$ of V which contains all ordinals of V and which is a model of ZFC must coincide with V, and therefore it gives us nothing new.

1.2 Discussion. Now we come to the concept of forcing. A forcing notion $P \in V$ is just a partially ordered set (not empty of course). Usually a partial order is required to satisfy $p \leq q \,\&\, q \leq p \Rightarrow p = q$, but we shall not (this is just a technicality), this is usually called pre-partial order or quasi order. It is also called a forcing notion. We normally assume that P as a minimal element denoted by $\emptyset_P$, i.e.

$$(\forall q \in P)(P \vDash \emptyset_P \leq q),$$

really from Chapter II on, we do not lose generality as by adding such a member we get an equivalent forcing notion, see §5. We want to add to V a subset G of P as follows.

(1) G is directed (i.e., every two members of G have an upper bound in G) and downward closed (i.e., if $x \leq y \in G$ then also $x \in G$).

Trivial examples of a set G which satisfies (1) is the empty set $\emptyset$ and $\{x : x \leq p\}$ for $p \in P$.

The following should be taken as a declaration of intent rather than an exactly formulated requirement.

(2) We want that $G \notin V$ and moreover G is "general" or "random" or "without any special property".

We aim at constructing a (transitive) set $V[G]$ which is a model of ZFC with the same ordinals as V, such that $V \subseteq V[G]$ and $G \in V[G]$, and which is minimal among the sets which satisfy these requirements.

So we can look at P as a set of approximations to G, each $p \in P$ giving some information on G, and $p \leq q$ means q gives more information; this view is helpful in constructing suitable forcing notions.

Where does the main problem in constructing such a set $V[G]$ lie? In the universe of set theory the ordinals of V are countable ordinals since V itself is countable. But an ordinal of V may be uncountable from the point of view of V (since V is a model of ZFC and the existence of uncountable ordinals is provable in ZFC). Since for each ordinal $\alpha \in V$ the information that α is countable is available outside V, G may contain i.e. code that information for each $\alpha \in V$. In this case every ordinal of V (and hence of $V[G]$) is countable in $V[G]$ and thus $V[G]$ cannot be a model of ZFC. How do we avoid this danger?

By choosing G to be "random" we make sure that it does not contain all that information.

While we choose a "random" G we do not aim for a random $V[G]$, but we want to construct a $V[G]$ with very definite properties. Therefore we can regard p as the assertion that $p \in G$ and as such p provides some information about G. All the members of G, taken together, give the complete information about G.

Now we come back to the second requirement on G and we want to replace the nebulous requirement above by a strict mathematical requirement.

1.3 Definition. (1) A subset $\mathcal{I}$ of P is said to be a *dense* subset of P if it satisfies

$$(\forall p \in P)(\exists q \in P)(p \leq q \,\&\, q \in \mathcal{I})$$

(2) Call $\mathcal{I} \subseteq P$ open (or upward closed) if for every $p, q \in P$

$$p \geq q \,\&\, q \in \mathcal{I} \Rightarrow p \in \mathcal{I}$$

1.4. Discussion. Since we want G to contain as many members of P as possible without contradicting the requirement that it be directed, we require:

$(2)'$ $G \cap \mathcal{I} \neq \emptyset$ for every dense open subset $\mathcal{I}$ of P which is in V.

1.4A Definition. A subset G of P which satisfies requirements (1) and $(2)'$ is called *generic* over V (we usually omit V), where this adjective means that G satisfies no special conditions in addition to those it has to satisfy.
The forcing theorem will assert that for a generic $G, V[G]$ is as we intended it to be.

Does $(2)'$ imply that $G \notin V$? Not without a further assumption, since if P consists of a single member p then $G = \{p\}$ satisfies (1) and $(2)'$ and $G \in V$. However if we assume that P has no trivial branch, in the sense that above every member of P there are two incompatible members, then indeed $G \notin V$ (incompatible means having no common upper bound). To prove this notice that if $G \in V$, then $P \setminus G$ is a dense open subset of P in V, remember that G is downward closed, and by $(2)'$ we would have $G \cap (P \setminus G) \neq \emptyset$, which is a contradiction.

1.5 The Forcing Theorem, Version A. (1) If G is a generic subset of P over V, then there is a transitive set $V[G]$ which is a model of ZFC, $V \subseteq V[G]$, $G \in V[G]$ and V and $V[G]$ have the same ordinals and we can allow V as a class of $V[G]$ (i.e. in the axioms guaranting (first order) definiable sets exists "$x \in V$" is allowed as a predicate).
(2) P has a generic subset G, moreover for every $p \in P$ there is a $G \subseteq P$ generic over V, $p \in G$. $\square_{1.5}$

1.6 Discussion. We shall not prove 1.5(1), but we shall prove 1.5(2). Since V is countable, P has at most $\aleph_0$ dense subsets in V; let us denote them with $\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2, \ldots$ we shall construct by induction a sequence p_n. We take an arbitrary p_0. We choose p_{n+1} so that $p_n \leq p_{n+1} \in \mathcal{I}_n$; this is possible since $\mathcal{I}_n$ is dense. We take $G = \{q \in P : \exists n(q \leq p_n)\}$. It is easy to check that this G is generic.

Since we want to prove theorems about $V[G]$ we want to know what are the members of $V[G]$. We cannot have in V full knowledge on all the members of $V[G]$ since this would cause these sets to belong to V. So we have to agree that we do not know the set G, but, as we want as much knowledge on $V[G]$ as possible, we require that except for that we have in V full knowledge of all members of $V[G]$, more specifically V contains a prescription for building that member out of G. We shall call these prescriptions "names". We shall be guided in the construction of the names by the idea that $V[G]$ contains only those members that it has to.

Remember:

1.7 Definition. We define the rank of any $a \in V$:

$$\mathrm{rk}(a) \text{ is } \bigcup\{\mathrm{rk}(b) + 1 : b \in a\}$$

(note if $a = \emptyset, \mathrm{rk}(a) = 0$), the union of a set of ordinals is an ordinal, hence $\mathrm{rk}(a)$ is an ordinal if defined, and by the axiom of regularly $\mathrm{rk}(a)$ is defined for every a. So

1.8 Definition. We define what is a P-name (or name for P or a name in P) τ of rank $\leq \alpha$, and what is its interpretation $\tau[G]$. If P is clear we omit it.

This is done by induction on α. τ is a name of rank $\leq \alpha$ if it has the form $\tau = \{(p_i, \tau_i) : i < i_0\}$, $p_i \in P$ and each τ_i is a name of some rank $< \alpha$.

The interpretation $\tau[G]$ of τ is $\{\tau_i[G] : p_i \in G, i < i_0\}$

1.9 Definition.

(1) Let $\mathrm{rk}_n(\tau) = \alpha$ if τ is a name (for some P) of rank $\leq \alpha$ but not a name of rank $\leq \beta$ for any $\beta < \alpha$.

(2) For $a \in V$ and forcing notion P, $\dot{a}$ is a P-name defined by induction on $\mathrm{rk}(a)$;

$$\dot{a} = \{(p, \dot{b}) : p \in P, b \in a\}$$

(3) $\underset{\sim}{G} = \{(p, \dot{p}) : p \in P\}$ (when necessary we denote it by ($\underset{\sim}{G}_P$).

(4) $\mathrm{rk}_r(\tau)$, the revised rank of a P-name τ is defined as follows: $\mathrm{rk}_r(\tau) = 0$ iff $\tau = \dot{a}$ for some $a \in V$

Otherwise

$$\mathrm{rk}_r(\tau) = \bigcup\{\mathrm{rk}_r(\sigma) + 1 : (p, \sigma) \in \tau \text{ for some } p\}$$

1.9A Remark. 1) Usually, we use $\underset{\sim}{\tau}, \underset{\sim}{f}, \underset{\sim}{a}$ etc. to denote P-names not necessarily of this form.

Eventually we lapse to denoting $\dot{a}$ (the P-name of a) by a, abusing our notation, in fact, no confusion arrives.

1.10 Claim. Given a forcing notion P, and $G \subseteq P$ generic over V, we have:

(1) $\mathrm{rk}_r(\tau) \leq \mathrm{rk}_n(\tau)$ and $\mathrm{rk}(\tau[G]) \leq \mathrm{rk}_n(\tau)$ for any P-name τ.

(2) for $a \in V$, $\dot{a}[G] = a$

(3) $\underset{\sim}{G}[G] = G$.

(4) $\text{rk}_r(\tau), \text{rk}_n(\tau)$ are well defined ordinals, for any P-name τ.

Proof. Trivial. $\square_{1.10}$

1.11 Discussion. Notice that while every name belongs to V, the values of the names are not necessarily in V since the definition of the interpretation of a name cannot be carried out in V. It turns out that these names are sufficient in the sense that the set of their values is a set $V[G]$ as required:

1.12 The Forcing Theorem (strengthened), Version B. In version A, in addition $V[G] = \{\tau[G] : \tau \in V$, and τ is a P-name $\}$. $\square_{1.12}$

We want to know which properties hold in $V[G]$. The properties we are interested in are the first order properties of $V[G]$, i.e., the properties given by formulas of the predicate calculus. We shall refer to the members of $V[G]$ by their names so we shall substitute the names in the formulas.

1.13 Definition. If $\tau_1 \dots, \tau_n$ are names, for the forcing notion P, $\varphi(x_1, \dots, x_n)$ a first-order formula of the language of set theory with an additional unary predicate for V, then we write $p \Vdash_P$ "$\varphi(\tau_1 \dots, \tau_n)$" ($p$ forces $\varphi(\tau_1 \dots, \tau_n)$ for the forcing P) *if* for every generic subset G of P which contains p we have:

$\varphi(\tau_1[G] \dots, \tau_n[G])$ is satisfied (=is true) in $V[G]$,

in symbols $V[G] \models$ "$\varphi(\tau_1[G], \dots, \tau_n[G])$".

1.14 The Forcing Theorem, Version C. If G is a generic subset of P *then* (in addition to the demands in versions A and B we have:) for every $\varphi(\tau_1 \dots, \tau_n)$ as above there is a $p \in G$ such that $p \Vdash_P$ "$\neg\varphi(\tau_1 \dots, \tau_n)$" or $p \Vdash_P$ "$\varphi(\tau_1 \dots, \tau_n)$". Therefore $V[G] \models$ "$\varphi(\tau_1[G] \dots, \tau_n[G])$" iff for some $p \in G$ $p \Vdash_P$ "$\varphi(\tau_1 \dots, \tau_n)$". Moreover $\Vdash$ (as a relation) is definable in V. $\square_{1.14}$

This is finally the version we shall actually use, but we shall not prove this theorem either.

The *forcing relation* $\Vdash_P$ clearly depends on P. If we deal with a fixed P we can drop the subscript P. We refer to P as the *forcing notion*.

The rest of the section is devoted to technical lemmas which will help to use the forcing theorem.

1.15 Definition. For $p, q \in P$ we say that p and q are *compatible* if they have an upper bound. $\mathcal{I} \subseteq P$ is an *antichain* if every two members of $\mathcal{I}$ are incompatible. $\mathcal{I} \subseteq P$ is a *maximal antichain* if $\mathcal{I}$ is an antichain and there is no antichain $\mathcal{J} \subseteq P$ which properly includes $\mathcal{I}$. We say $\mathcal{I} \subseteq P$ is pre-dense (above $p \in P$) if for every $q \in P$ $(q \geq p)$ some $q^\dagger \in \mathcal{I}$ is compatible with q. We say $\mathcal{I} \subseteq P$ is dense above $p \in P$ if for every $q \in P$ such that $q \geq p$ there is r, $q \leq r \in \mathcal{I}$; we may omit "above p". We define "$\mathcal{I} \subseteq P$ is pre-dense above $p \in P$" similarly.

1.16 Lemma. Let G be a downward closed subset of P. Then: G is generic (over V) *iff* for every maximal antichain $\mathcal{I} \in V$ of P we have $|G \cap \mathcal{I}| = 1$.

Proof. Suppose G is generic. Since G is directed it cannot contain two incompatible members and hence $|G \cap \mathcal{I}| \leq 1$. Given $\mathcal{I} \in V$, a subset of P, let $\mathcal{J} = \{p \in P : (\exists q \in \mathcal{I}) p \geq q\} \in V$, i.e., $\mathcal{J}$ is the upward closure of $\mathcal{I}$. So $\mathcal{J}$ is obviously upward closed i.e. an open subset, we shall now show that if $\mathcal{I}$ is a maximal antichain of P, then $\mathcal{J}$ is dense. For any $r \in P$ clearly r is compatible with some member q of $\mathcal{I}$ (otherwise $\mathcal{I} \bigcup \{r\}$ would be an antichain properly including the maximal antichain $\mathcal{I}$), let $p \geq r, q$. Then, by the definition of $\mathcal{J}$, $p \in \mathcal{J}$ and we have proved the density of $\mathcal{J}$.
Since $\mathcal{J}$ is dense and open by Definition 1.4A we know $G \cap \mathcal{J} \neq \emptyset$, let $p \in G \cap \mathcal{J}$. Since $p \in \mathcal{J}$, there is a $q \in \mathcal{I}$ such that $q \leq p$, and since $p \in G$ and G is generic, $q \in G$ and so $q \in G \cap \mathcal{I}$, hence $|G \cap \mathcal{I}| \geq 1$. So (assuming $G \subseteq P$ is generic over V) we have proved: for every maximal antichain $\mathcal{I} \in V$ of P, $|G \cap \mathcal{I}| = 1$, thus proving the only if part of the lemma.

Now assume that for every maximal antichain $\mathcal{I} \in V$ we have $|G \cap \mathcal{I}| = 1$. First let $\mathcal{J} \in V$ be a dense subset of P and we shall prove $G \cap \mathcal{J} \neq \emptyset$. By Zorn's lemma there is an antichain $\mathcal{I} \subsetneq \mathcal{J}$ which is maximal among the antichains in $\mathcal{J}$, i.e. the antichains of P which are subsets of $\mathcal{J}$. We claim that $\mathcal{I}$ is a maximal antichain. Let $r \in P$, we have to prove that r is compatible with some member of $\mathcal{I}$ (and hence $\mathcal{I}$ cannot be properly extended to an antichain). Since $\mathcal{J}$ is dense there is a $p \in \mathcal{J}$ such that $p \geq r$. Since $p \in \mathcal{J}$ and $\mathcal{I}$ is an antichain maximal in $\mathcal{J}$ necessarily p is compatible with some member q of $\mathcal{I}$, hence r is also compatible with q; so we have finished proving "$\mathcal{I}$ is a maximal antichain of P". So by our present assumption $|G \cap \mathcal{I}| = 1$ hence $G \cap \mathcal{J} \supseteq G \cap \mathcal{I} \neq \emptyset$.

Secondly to see that G is directed let $q, r \in G$ and let $\mathcal{J} = \{p \in P : p \geq q, r$ or p is incompatible with q or p is incompatible with $r\}$. Clearly $\mathcal{J} \in V$, to prove that $\mathcal{J}$ is dense let $s \in P$. If s is incompatible with q then $s \in \mathcal{J}$. Otherwise there is a $t \in P$ such that $s, q \leq t$. If t is incompatible with r then $t \in \mathcal{J}$, and we know that $t \geq s$. Otherwise there is a $w \in p$ such that $w \geq t, r$. Since $t \geq s, q$ we have $w \geq q, r$ and hence $w \in \mathcal{J}$. Since $w \geq t \geq s$ we know $\mathcal{J}$ is dense. By what we have shown above, $G \cap \mathcal{J} \neq \emptyset$. Let $p \in G \cap \mathcal{J}$. We shall see that p cannot be incompatible with q or with r, therefore, since $p \in \mathcal{J}$, $p \geq q, r$. We still have to prove that no two members of G, such as p and q, are incompatible. Suppose $p, q \in G$ and p and q are incompatible. We extend the antichain $\{p, q\}$, by Zorn's lemma to a maximal antichain $\mathcal{I} \in V$. We have $\mathcal{I} \cap G \supseteq \{p, q\}$, contradicting $|\mathcal{I} \cap G| = 1$.

As part of the assumption of 1.16 is "$G \subseteq P$ is downward closed", and we have proved G is directed, and [$\mathcal{J} \in V$ is a dense subset of $P \Rightarrow G \cap \mathcal{J} \neq \emptyset$], we have proved that G is a generic subset of P over V (see Definition 1.4A). Hence we have finished proving also the if part of the lemma. $\square_{1.16}$

1.17 Lemma. If $\mathcal{J}$ is a pre-dense subset of P in V and G is a generic subset of P *then* $G \cap \mathcal{J} \neq \emptyset$.

Proof. Let $\mathcal{J}^\dagger = \{p \in P : (\exists q \in \mathcal{J}) p \geq q\}$. Let us prove that $\mathcal{J}^\dagger$ is a dense open subset of P. Now $\mathcal{J}^\dagger$ is obviously upward-closed. Let $r \in P$. Since $\mathcal{J}$ is pre-dense there is a $q \in \mathcal{J}$ such that q is compatible with r. Therefore, there is a $p \in P$ such that $p \geq q, r$. By the definition of $\mathcal{J}^\dagger$ we have $p \in \mathcal{J}^\dagger$. Thus we

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have proved that for every $r \in P$ there is a $p \in \mathcal{J}^\dagger$ such that $p \geq r$, and so $\mathcal{J}^\dagger$ is dense. Since $\mathcal{J} \in V$ and $\mathcal{J}^\dagger$ is constructed from $\mathcal{J}$ in V we have $\mathcal{J}^\dagger \in V$. Since G is generic over V we have $G \cap \mathcal{J}^\dagger \neq \emptyset$. Let $p \in G \cap \mathcal{J}^\dagger$. By the definition of $\mathcal{J}^\dagger$ there is a $q \in \mathcal{J}^\dagger$ such that $q \leq p$. Since G is downward closed we have $q \in G$ and hence $q \in G \cap \mathcal{J} \neq \emptyset$, which is what we had to prove. $\square_{1.17}$

1.18 Lemma. Let $q \in P$, and let $\mathcal{I}$ be a subset of P in V which is pre-dense above q. For every generic subset G of P if $q \in G$ then $G \cap \mathcal{I} \neq 0$.

Proof. Let $\mathcal{I}^\dagger = \mathcal{I} \bigcup \{p \in P : p \text{ is incompatible with } q\}$. Since $\mathcal{I} \in V$ also $\mathcal{I}^\dagger \in V$. Let us prove that $\mathcal{I}^\dagger$ is a pre-dense subset of P. Let $r \in P$. If r is incompatible with q then $r \in \mathcal{I}^\dagger$. If r is compatible with q then there is an $s \in P$ such that $s \geq r, q$. Since $\mathcal{I}$ is pre-dense above q, necessarily s is compatible with some member of $\mathcal{I}$, and hence r is compatible with the same member of $\mathcal{I}$ which neccessarily is also in $\mathcal{I}^\dagger$. Thus we have shown that $\mathcal{I}^\dagger$ is pre-dense. Let G be a generic subset of P such that $q \in G$. Since $\mathcal{I}^\dagger$ is pre-dense and $\mathcal{I}^\dagger \in V$ we have $G \cap \mathcal{I}^\dagger \neq \emptyset$. Let $t \in G \cap \mathcal{I}^\dagger$. Since $t, q \in G$, t is compatible with q, hence by the definition of $\mathcal{I}^\dagger$ we must have $t \in \mathcal{I}$ and thus $t \in G \cap \mathcal{I} \neq \emptyset$. $\square_{1.18}$

1.19 Lemma. Let $\mathcal{I} = \{p_i : i < i_0\}$ be an antichain in P and $\{\tau_i : i < i_0\}$ a corresponding indexed family of P-names (in V). *Then* there is a name τ such that: for every $i < i_0$ and for every generic G, if $p_i \in G$ then $\tau[G] = \tau_i[G]$ (and $\tau[G] = \emptyset$ if $G \cap \{p_i : i < i_0\} = \emptyset$). (We recall that a generic G contains at most one member of $\mathcal{I}$ and if $\mathcal{I}$ is a maximal antichain of P then G contains exactly one member of $\mathcal{I}$).

1.19A Remark. This means we can define a name by cases.

Proof. Suppose $\tau_i = \{\langle p_{i,j}, \tau_{i,j}\rangle : j < j_i\}$, (of course $j_i = 0$ is possible) and let $\tau = \{\langle r, \tau_{i,j}\rangle : j < j_i,\ i < i_0, r \geq p_{i,j}$ and $r \geq p_i\}$. $\square_{1.19}$

We note also: