CAMBRIDGE

Cambridge University Press
978-1-107-16835-0 — Constructibility
Keith J. Devlin
Excerpt
More Information

Chapter I
Preliminaries

The fundamental set theory of this book is Zermelo-Fraenkel set theory. In this chapter we give a brief account of this theory, insofar as we need it. Sections 1 through 5 cover the early development of the theory up to ordinal and cardinal numbers. The remaining six sections deal with some special topics of direct relevance to the subject matter of this book, and the coverage is therefore a little more complete than in the previous sections.

1. The Language of Set Theory

The language of set theory, LST, is the first-order language with predictates $=$ (equality) and $\in$ (set membership), logical symbols $\wedge$ (and), $\neg$ (not), and $\exists$ (there exists), variables $v_0, v_1, \ldots,$ and (for convenience) brackets $(,)$.

The *primitive* (or *atomic*) *formulas* of LST are strings of the forms

$$(v_m = v_n), \quad (v_m \in v_n).$$

The *formulas* of LST are generated from the primitive formulas by means of the following schemas: if Φ, Ψ are formulas, so too are the strings

$$(\Phi \wedge \Psi), \quad (\neg \Phi), \quad (\exists\, v_n \Phi).$$

(We generally use capital Greek letters to denote formulas of LST.)

The notions of *free* and *bound* variables are defined as usual. A *sentence* is a formula with no free variables.

We write $x \notin y$ for $\neg\,(x \in y)$ and $x \neq y$ for $\neg\,(x = y)$. (We generally use, x, y, z, etc. to denote arbitrary variables of LST.)

The defined logical symbols $\vee$, $\rightarrow$, $\leftrightarrow$, $\forall$ are introduced in the usual way, and are frequently treated as if they were basic symbols of LST (i.e. having the same status as $\wedge$, $\neg$, $\exists$). Likewise for the bounded quantifiers $(\exists\, v_m \in v_n)$ and $(\forall\, v_m \in v_n)$ (where $m \neq n$), introduced by the schemas:

$$(\exists\, v_m \in v_n)\, \Phi \quad \text{replaces} \quad \exists\, v_m((v_m \in v_n) \wedge \Phi);$$

$$(\forall\, v_m \in v_n)\, \Phi \quad \text{replaces} \quad \forall\, v_m((v_m \in v_n) \rightarrow \Phi).$$

© in this web service Cambridge University Press www.cambridge.org

CAMBRIDGE

Cambridge University Press
978-1-107-16835-0 — Constructibility
Keith J. Devlin
Excerpt
More Information

The symbols $\subseteq$ and $\exists!$ are defined thus:

$y \subseteq z$ abbreviates $(\forall x \in y)(x \in z)$;

$\exists! x \Phi$ abbreviates $\exists y \forall x (y = x \leftrightarrow \Phi)$.

(Thus $\exists! x \Phi$ means "there is a unique x such that Φ".) We also write

$y \subset z$ to mean $y \subseteq z \wedge y \neq z$.

The above abbreviations are never regarded as a fundamental part of the language LST, however, unlike the bounded quantifiers, etc.

One final remark. In writing formulas, we strive for legibility at the expense of strict adherence to the syntax of LST. This particularly applies to our use of parentheses, which are omitted wherever possible. Also, when nesting of clauses is required, we sometimes use both (square) brackets as well as parentheses, for clarity. Out notation for the interpretation of variables in formulas is also chosen with clarity in mind. If we write, say, $\Phi(v_i, v_j)$, we mean that the free variables of Φ are amongst the variables v_1, v_j. If we subsequntly write $\Phi(x, y)$, where x and y are specific sets, we mean that Φ is a valid assertion when x interprets v_i and y interprets v_j. (Of course, we have also decided to use x, y, z, etc. to denote arbitrary variables of LST. But in any given case, the context should indicate the intended meaning.[1]

2. *The Zermelo-Fraenkel Axioms*

The theory ZF is the LST theory whose axioms are the usual axioms for first-order logic (for the langugage LST), together with the following axioms (i)–(vii):

(i) Extensionality: $\forall x \forall y [\forall z (z \in x \leftrightarrow z \in y) \rightarrow (x = y)]$
(ii) Union: $\forall x \exists y \forall z [z \in y \leftrightarrow (\exists u \in x)(z \in u)]$
(iii) Infinity: $\exists x [\exists y (y \in x) \wedge (\forall y \in x)(\exists z \in x)(y \in z)]$
(iv) Power Set: $\forall x \exists y \forall z [z \in y \leftrightarrow z \subseteq x]$
(v) Foundation: $\forall x [\exists y (y \in x) \rightarrow \exists y (y \in x \wedge (\forall z \in y)(z \notin x))]$
(vi) Comprehension (schema): $\forall \vec{a} \forall x \exists y \forall z [z \in y \leftrightarrow z \in x \wedge \Phi(z, \vec{a})]$,

where Φ is any LST formula whose free variables are amongst z, $\vec{a}$, and where the variables $\vec{a}$, x, y, z are all distinct.

(We use $\vec{x}$, $\vec{a}$, etc. to denote finite strings of variables, $\forall \vec{a}$ to abbreviate $\forall a_1, \ldots, \forall a_n$ and $\Phi(z, \vec{a})$ to abbreviate $\Phi(z, a_1, \ldots, a_n)$. In more complicated situations,

1 Strictly speaking there is no clash of notation here. As far as formal set theory is concerned there are simply *variables* (to denote "sets"). But as usual, to avoid incomprehensible use of quantifiers and formulas to define specific sets, we argue in a loose, *semantic* fashion whenever possible, and then it can be useful to distinguish between "formal variables" and "sets which interpret those variables".

© in this web service Cambridge University Press www.cambridge.org

CAMBRIDGE

Cambridge University Press
978-1-107-16835-0 — Constructibility
Keith J. Devlin
Excerpt
More Information

we often use expressions such as $\vec{x}_0, \ldots, \vec{x}_n$. Here, $\vec{x}_0$ will denote some sequence $x_{00}, \ldots, x_{0k}$, $\vec{x}_1$ will be another sequence $x_{10}, \ldots, x_{1l}$, possibly of a different length, according to context, and so on.)

(vii) Collection (schema):

$$\forall \vec{a}\,[\forall x\,\exists y\,\Phi(y, x, \vec{a}) \rightarrow \forall u\,\exists v(\forall x \in u)(\exists y \in v)\,\Phi(y, x, \vec{a})],$$

where Φ is any LST-formula whose free variables are amongst y, x, $\vec{a}$, and where the variables $\vec{a}$, x, y, u, v are all distinct.

In (iii), the exact formulation of the Axiom of Infinity is not important, and different texts often give different formulations. The main point is to guarantee the existence of at least one infinite set. Axiom (vi) (the Comprehension Axiom schema) is sometimes referred to as the Subset Selection schema. The German word *Aussonderungsaxiom* is also quite common for this axiom scheme. In Axiom (vii) (Collection), notice that we have placed the variable y before the variable x. This is purely a stylistic convention, of course, and reflects the fact that in our representation of a function as a set of ordered pairs, we shall take the first member of each ordered pair as the value of the function and the second element as the argument. Axiom schemas (vi) and (vii) are often replaced by a single schema: the *Axiom of Replacement*.

Notice that by virtue of the two axiom schemas, the above list of axioms for ZF is infinite. We shall soon be able to prove that no finite collection of LST sentences suffices to axiomatise ZF.

By the Axiom of Infinity, there exists at least one set. The Axiom of Comprehension then yields the existence of the empty set $\emptyset$. Many texts include as an axiom of ZF the *Null Set Axiom*, which is the assertion that there exists a set having no elements, viz.:

$$\exists x\,\forall y\,(y \notin x).$$

Zermelo-Fraenkel set theory includes one further axiom:

(viii) Axiom of Choice (AC):

$$\forall x\,[(\forall y \in x)(y \neq \emptyset) \wedge (\forall y, y' \in x)(y \neq y' \rightarrow \forall w(w \in y \leftrightarrow w \notin y') \rightarrow (\exists z)(\forall y \in x)(\exists! v \in y)(v \in z)].$$

We denote Zermelo-Fraenkel set theory (which includes AC) by ZFC. This nomenclature is now fairly standard, despite the rather unfortunate fact that it means that the letters ZF do not stand for "Zermelo-Fraenkel" set theory, but just a part of that theory. To try to avoid any confusion, throughout the book we shall stick to the abbreviated notations ZF and ZFC. Hence, we shall have the "equation"

$$\text{ZFC} = \text{ZF} + \text{AC}.$$

© in this web service Cambridge University Press www.cambridge.org

CAMBRIDGE

Cambridge University Press
978-1-107-16835-0 — Constructibility
Keith J. Devlin
Excerpt
More Information

ZFC is our basic set theory. On occasions it will be important to note that AC is not being used in an argument, and in such cases we shall write, for example,

$$\mathrm{ZF} \vdash \Phi$$

or else

$$\Phi \to_{\mathrm{ZF}} \Psi$$

to mean, respectively, that Φ is provable in ZF or that Ψ is provable from Φ together with the axioms of ZF.

3. *Elementary Theory of* ZFC

3.1 (Sets and Classes). The basic objects of discussion of ZFC (i. e. the objects over which the variables range) are called *sets*. The *universe* is the collection of all sets, and is denoted by V. If $\Phi(v_0, v_1, \ldots, v_n)$ is an LST formula and $x_1, \ldots, x_n$ are sets, the collection of all sets x for which $\Phi(x, x_1, \ldots, x_n)$ is a *class*, denoted by

$$\{x \mid \Phi(x, x_1, \ldots, x_n)\}.$$

Every set, y, is a class (consider the formula $\Phi(x, y) \equiv (x \in y)$), but not every class is a set (consider the formula $\Phi(x) \equiv (x \notin x)$, which would lead at once to the Russell paradox if the class it defined were a set). We often write

$$\{x \in y \mid \Phi(x, x_1, \ldots, x_n)\}$$

in place of

$$\{x \mid x \in y \wedge \Phi(x, x_1, \ldots, x_n)\}.$$

(By the Axiom of Comprehension, this class is always a set.) We generally use capital Roman letters X, Y, Z etc. to denote classes, with lower case Roman letters being reserved for sets (as well as for variables of LST, which denote sets, of course). A class which is not a set is called a *proper class*. Proper classes do not fall under the scope of the axioms of ZFC, but their usage is convenient. We assume the reader is familiar both with the use of proper classes in set theory and the means by which such usage may be avoided if required. A particular example occurs in VI.1, where we discuss the rudimentary functions. It is convenient, though avoidable, to develop the relevant theory in terms of "functions" defined on the whole of V, even though, as proper classes these cannot be *functions* in the sense of set theory at all.

Our set-theoretic notation is standard. The set consisting of precisely the elements $x_1, \ldots, x_n$ is denoted by

$$\{x_1, \ldots, x_n\};$$

© in this web service Cambridge University Press www.cambridge.org

CAMBRIDGE

Cambridge University Press
978-1-107-16835-0 — Constructibility
Keith J. Devlin
Excerpt
More Information

$\{x\}$ is the *singleton* of x, and $\{x, y\}$ is the *unordered pair* of x, y. Many texts include as an axiom of ZF the *Pairing Axiom*, which asserts that for every pair of elements x, y, the set $\{x, y\}$ exists, i.e.

$$\forall x \forall y \exists z \forall u (u \in z \leftrightarrow u = x \vee u = y).$$

However, as this "axiom" is easily proved from the axioms we listed earlier, we did not take it as a basic axiom.

The *ordered pair* of x and y is defined by

$$(x, y) = \{\{x\}, \{x, y\}\},$$

and has the property that

$$(x, y) = (x', y') \quad \text{iff } x = x' \text{ and } y = y'.$$

The *union* of x (i.e. the set of all members of all members of x) is denoted by $\bigcup x$, and is guaranteed to exist by the Union Axiom. We write $x \cup y$ instead of $\bigcup \{x, y\}$. The *intersection* of x, $\bigcap x$, is defined by

$$y \in \bigcap x \quad \text{iff } (\forall z \in x)(y \in z),$$

and is a set whenever $x \neq \emptyset$. (By our definition, $\bigcap \emptyset = V$, but this is not a case that will ever concern us.) We write $x \cap y$ for $\bigcap \{x, y\}$. The *difference* of x and y is defined by

$$x - y = \{z \in x \mid z \notin y\}.$$

The *power set* of x (i.e. the set of all subsets of x) is denoted by $\mathscr{P}(x)$, and is guaranteed to exist by the Power Set Axiom.

3.2 (Ordinals). A class M is said to be *transitive* if

$$x \in y \in M \rightarrow x \in M.$$

If $\text{Trans}(v_0)$ denotes the LST formula

$$(\forall v_1 \in v_0)(\forall v_2 \in v_1)(v_2 \in v_0),$$

then a set x will be transitive iff $\text{Trans}(x)$.

An *ordinal number* (or simply, an *ordinal*) is a transitive set which is linearly ordered by $\in$. We use $\alpha, \beta, \gamma, \ldots$ to denote ordinals. We denote by $\text{On}(v_0)$ the LST-formula

$$\text{Trans}(v_0) \wedge (\forall v_1 \in v_0)(\forall v_2 \in v_0)(v_1 = v_2 \vee v_1 \in v_2 \vee v_2 \in v_1).$$

It is not hard to show that a set x will be an ordinal iff $\text{On}(x)$.

© in this web service Cambridge University Press www.cambridge.org

CAMBRIDGE

Cambridge University Press
978-1-107-16835-0 — Constructibility
Keith J. Devlin
Excerpt
More Information

If α, β are ordinals, either $\alpha = \beta$ or $\alpha \in \beta$ or $\beta \in \alpha$. So the class

$$\mathrm{On} = \{x \mid \mathrm{On}(x)\}$$

is totally ordered by $\in$. We often write $\alpha < \beta$ instead of $\alpha \in \beta$, and $\alpha \leqslant \beta$ instead of $(\alpha < \beta \vee \alpha = \beta)$. It is easily seen that $\alpha < \beta$ is equivalent to $\alpha \subset \beta$. Moreover, for any ordinal α,

$$\alpha = \{\beta \mid \beta < \alpha\}.$$

By the Axiom of Foundation, the relation $<$ is in fact a well-ordering of On (i.e. every non-empty subset of On has a $<$-least element).

If A is a set of ordinals, then $\cup A$ is also an ordinal. In fact, $\cup A$ is the least ordinal δ such that $(\forall \alpha \in A)(\alpha \leqslant \delta)$. This least δ is also called the *supremum* of A, denoted by $\sup(A)$. Thus $\sup(A)$ and $\cup A$ coincide.

The first ordinal (under the canonical well-ordering $\in$) is the null set, $\emptyset$, but when considered as an ordinal it is usually denoted by 0. The next ordinal is the set $\{0\}$, denoted by 1. Then comes the ordinal $\{0, 1\}$, denoted by 2, followed by $3 = \{0, 1, 2\}$, and so on. If α is an ordinal, so too is $\alpha \cup \{\alpha\}$, and there is no ordinal γ strictly between α and $\alpha \cup \{\alpha\}$. We call $\alpha \cup \{\alpha\}$ the *successor* of α, denoted by $\alpha + 1$. Any ordinal of the form $\alpha + 1$ is called a *successor ordinal*. An ordinal α is a successor ordinal iff $\mathrm{succ}(\alpha)$, where $\mathrm{succ}(v_0)$ is the LST-formula

$$\mathrm{On}(v_0) \wedge (\exists v_1 \in v_0)(\forall v_2 \in v_0)(v_2 \in v_1 \vee v_2 = v_1).$$

A non-zero ordinal which is not a successor ordinal is called a *limit ordinal*. If $\lim(v_0)$ is the LST-formula

$$\mathrm{On}(v_0) \wedge (\exists v_1 \in v_0)(v_1 = v_1) \wedge (\forall v_1 \in v_0)(\exists v_2 \in v_0)(v_1 \in v_2),$$

then an ordinal α will be a limit ordinal iff $\lim(\alpha)$. Using the Axiom of Infinity, together with other ZF axioms, it can be shown that a limit ordinal exists. The least limit ordinal is denoted by ω. The elements of the set ω are precisely the finite ordinal numbers, and are called the *natural numbers*. We usually denote natural numbers by m, n, i, j, k, etc. Notice that ω is definable by the formula

$$\lim(v_0) \wedge (\forall v_1 \in v_0)(\mathrm{succ}(v_1) \vee (\forall v_2 \in v_1)(v_2 \neq v_2)).$$

We usually write $\exists \alpha \Phi(\alpha)$ in place of

$$\exists v_0 [\mathrm{On}(v_0) \wedge \Phi(v_0)],$$

and $\forall \alpha \Phi(\alpha)$ in place of

$$\forall v_0 [\mathrm{On}(v_0) \rightarrow \Phi(v_0)].$$

If $(X, <)$ is a well-ordered set, there is a unique ordinal number α such that $(X, <)$ is isomorphic to α (with the usual ordering). This α is called the *order-type* of $(X, <)$, denoted by $\mathrm{otp}(X, <)$.

© in this web service Cambridge University Press www.cambridge.org

CAMBRIDGE

Cambridge University Press
978-1-107-16835-0 — Constructibility
Keith J. Devlin
Excerpt
More Information

3.3 (Relations and Functions). Let $n > 0$ be a natural number. The *n-tuple* $(x_1, \ldots, x_n)$ of sets $x_1, \ldots, x_n$ is defined thus:

$$\text{if } n = 1, \qquad (x_1) = x_1;$$

$$\text{if } n > 1, \qquad (x_1, \ldots, x_n) = (x_1, (x_2, \ldots, x_n))$$
$$(= (x_1, (x_2, (x_3, \ldots, x_n))) = \text{etc.})$$

If $X_1, X_2, \ldots, X_n$ are classes, their *Cartesian product* is the class:

$$X_1 \times X_2 \times \ldots \times X_n = \{(x_1, \ldots, x_n) \mid x_1 \in X_1 \wedge \ldots \wedge x_n \in X_n\}.$$

We write X^2 in place of $X \times X$, X^3 for $X \times X \times X$, etc.

Let X be any class. An *n-ary relation* on X is a class $R \subseteq X^n$. We often write $R(\vec{x})$ in place of $(\vec{x}) \in R$.

Suppose R is an $(n+1)$-ary relation on a class X, where $n > 0$. The *domain* of R is the class

$$\text{dom}(R) = \{(\vec{x}) \mid \exists\, y R(y, \vec{x})\}.$$

The *range* of R is the class

$$\text{ran}(R) = \{y \mid \exists\, \vec{x}\, R(y, \vec{x})\}.$$

If $Z \subseteq X$, we set

$$R \restriction Z = \{(y, \vec{x}) \in R \mid \vec{x} \in Z\}.$$

(Notice that accoding to our conventions concerning finite strings of variables, $\vec{x} \in Z$ means $x_1 \in Z, \ldots, x_n \in Z$. If we want to mean that $(x_1, \ldots, x_n) \in Z$ we would write $(\vec{x}) \in Z$.)

We define

$$R'' Z = \text{ran}(R \restriction Z).$$

Let X be a class, $n > 0$. An *n-ary function* over X is an $(n+1)$-ary relation R on X such that

$$(\forall\, (\vec{x}) \in \text{dom}(R))(\exists!\, y) R(y, \vec{x}).$$

We often write $R(\vec{x}) = y$ instead of $R(y, \vec{x})$ in such cases. Thus $R(\vec{x})$ is the unique y such that $R(y, \vec{x})$. We say that R is *total* on X iff $\text{dom}(R) = X$.

Let f be an n-ary function over V. We write

$$f: X \to Y$$

to denote that (f is a function and) $\text{dom}(f) = X$ and $\text{ran}(f) \subseteq Y$. We say that f is *one-one* (or *injective*), and write

$$f: X \xrightarrow{(1-1)} Y,$$

© in this web service Cambridge University Press www.cambridge.org

CAMBRIDGE

Cambridge University Press
978-1-107-16835-0 — Constructibility
Keith J. Devlin
Excerpt
More Information

iff for all $x_1, x_2 \in X$,

$$x_1 \neq x_2 \rightarrow f(x_1) \neq f(x_2).$$

We say f is *onto* Y (or is a *surjection* to Y), and write

$$f\colon X \xrightarrow{\text{onto}} Y,$$

iff $\text{ran}(f) = Y$. We say f is a *bijection* iff it is both one-one and onto, and write

$$f\colon X \leftrightarrow Y.$$

If f is bijective there is a unique function $f^{-1}\colon Y \rightarrow X$ (called the *inverse* of f) such that

$$(\forall x \in X)(f^{-1}(f(x)) = x),$$

$$(\forall y \in Y)(f(f^{-1}(y)) = y).$$

Regardless of whether or not f is bijective, we set, for any $Z \subseteq Y$,

$$f^{-1}{}''Z = \{x \in X \mid f(x) \in Z\}.$$

The set $f^{-1}{}''Z$ is called the *preimage of* Z *under* f.

Notice that by our definition of the n-tuples $(x_1, \ldots, x_n)$, every function is a set of ordered pairs, regardless of whether or not the function is unary.

If X and Y are structures of the same type, we write

$$f\colon X \cong Y$$

if f is a bijection from X to Y which preserves the structure (i.e. if f is an *isomorphism*).

We denote the composition of functions f, g by $f \circ g$, as usual. Thus if $f\colon Y \rightarrow Z$ and $g\colon X \rightarrow Y$, we define $f \circ g\colon X \rightarrow Z$ by

$$(\forall x \in X)(f \circ g(x) = f(g(x))).$$

For any sets x, y we define

$${}^{x}y = \{f \mid f\colon x \rightarrow y\}.$$

The *identity function* is the unary function

$$\text{id} = \{(y, x) \mid y = x\}.$$

Of course, being a proper class, id is not strictly speaking a function at all, but for any set X, $\text{id} \restriction X$ will be a function, so this definition is convenient.

© in this web service Cambridge University Press www.cambridge.org

CAMBRIDGE

Cambridge University Press
978-1-107-16835-0 — Constructibility
Keith J. Devlin
Excerpt
More Information

A function whose domain is an ordinal is called a *sequence*; if α is that ordinal domain, we say that the sequence is an α-sequence. If f is an α-sequence, and if $f(\xi) = x_\xi$ for all $\xi < \alpha$, we often write $f = (x_\xi \mid \xi < \alpha)$.

If $f\colon I \to V$, and if we denote $f(i)$ by x_i for each $i \in I$, we often write $\{x_i \mid i \in I\}$ in place of $f'' I$, $\bigcup_{i \in I} x_i$ in place of $\cup (f'' I)$, and $\bigcap_{i \in I} x_i$ in place of $\cap (f'' I)$. Similarly, given a sequence $f = (x_\nu \mid \nu < \tau)$, we sometimes write $\bigcup_{\nu < \tau} x_\nu$ for $\cup (f'' \tau)$ and $\bigcap_{\nu < \tau} x_\nu$ for $\cap (f'' \tau)$. And if $h = (\alpha_\nu \mid \nu < \tau)$ is a sequence of ordinals, we would write $\sup_{\nu < \tau} \alpha_\nu$ for $\sup (h'' \tau)$.

The inverse functions to the ordered pair function are defined thus:

$$\text{if } u = (x, y), \text{ then } (u)_0 = x \text{ and } (u)_1 = y;$$

$$\text{if } u \text{ is not an ordered pair, then } (u)_0 = (u)_1 = \emptyset.$$

Similarly we define inverse functions $(u)^n_0, \ldots, (u)^n_{n-1}$ to the n-tuple function.

3.4 (Induction and Recursion). By using the Axiom of Foundation (together with other axioms of ZF), every instance of the following *schema of proof* by $\in$-*induction* can be proved in ZF:

$$\forall x [(\forall y \in x)\, \Phi(y) \to \Phi(x)] \to \forall x\, \Phi(x).$$

More generally, if X is any class, a relation $R \subseteq X^2$ is said to be *well-founded* iff:

(i) $(\forall x \in \mathrm{dom}(R))[\{y \mid R(y, x)\} \text{ is a set}]$;
(ii) $\forall a [a \neq \emptyset \wedge a \subseteq X \to (\exists x \in a)(\forall y \in a) \neg R(y, x)]$.

If R is such a relation, then every instance of the following *schema of proof by induction on* R is a theorem of ZF:

$$(\forall x \in X)[(\forall y \in X)(R(y, x) \to \Phi(y)) \to \Phi(x)] \to (\forall x \in X)\, \Phi(x).$$

It follows from the above that every instance of the following *schema of definition by recursion* is provable in ZF. Let G be a total $(n + 2)$-ary function over V, and let H be a total unary function over V such that the relation $\{(z, y) \mid z \in H(y)\}$ is well-founded. Then there is a unique, total $(n + 1)$-ary function F over V such that

$$F(y, \vec{a}) = G(y, \vec{a}, F \restriction (H(y) \times \{(\vec{a})\}))$$

(Actually, some care is required in formulating this result precisely. Given formulas which determine G and H, possibly with reference to certain set parameters, one can explicitly write down a third formula such that in ZF it is provable that the class determined by this formula has all of the properties required of F above. We assume the reader is quite familiar with all of this, though in fact we shall not really need to know the exact formulation. As is usually the case in set theory, all that we require is the knowledge that ZF allows definitions "of a recursive nature".)

© in this web service Cambridge University Press www.cambridge.org

CAMBRIDGE
Cambridge University Press
978-1-107-16835-0 — Constructibility
Keith J. Devlin
Excerpt
More Information

A particular case of the above recursion principle is when $H = \mathrm{id}$, when it is called the *principle of* $\in$*-recursion*. There are also recursion principles applicable to functions defined not on all of V but on some class, the most common example being when the class concerned is the class of all ordinals; $\in$-recursion restricted to the ordinals is known as *ordinal recursion*, and will be used frequently in this book.

The total, unary function TC (*transitive closure*) is defined by the $\in$-recursion

$$\mathrm{TC}(x) = x \cup \bigcup \{\mathrm{TC}(y) \mid y \in x\}.$$

(Intuitively, $\mathrm{TC}(x) = x \cup (\bigcup x) \cup (\bigcup \bigcup x) \cup (\bigcup \bigcup \bigcup x) \cup \ldots$. It is not hard to show that $\mathrm{TC}(x)$ is the $\subseteq$-smallest transitive set y such that $x \subseteq y$. We call the set $\mathrm{TC}(x)$ the *transitive closure* of x.

The relation $\{(x, y) \mid x \in \mathrm{TC}(y)\}$ is well-founded. Hence we can carry out definitions by recursion on this relation. This form of recursive definition will also be quite common in this book.

3.5 (The Cumulative Hierarchy). The *cumulative hierarchy of sets* is defined by the ordinal recursion

$$V_0 = \emptyset;$$
$$V_\alpha = \bigcup \{\mathscr{P}(V_\beta) \mid \beta < \alpha\}.$$

It is not hard to see that

$$V = \bigcup \{V_\alpha \mid \alpha \in \mathrm{On}\}.$$

(The proof makes central use of the Axiom of Foundation, and indeed the above "equation" may be taken as an alternative formulation of this axiom.)

The *rank* of a set x is the least ordinal α such that $x \in V_{\alpha+1}$. We may define the rank function directly by means of the $\in$-recursion

$$\mathrm{rank}\,(\emptyset) = 0$$
$$\mathrm{rank}\,(x) = \bigcup \{\mathrm{rank}\,(y) + 1 \mid y \in x\}.$$

Notice that $x \in y$ implies $\mathrm{rank}\,(x) < \mathrm{rank}\,(y)$.

4. Ordinal Numbers

The notion of an ordinal number plays a central role in set theory, and we have referred to ordinals several times already. In this section we consider, very briefly, the arithmetic of ordinal numbers.

© in this web service Cambridge University Press www.cambridge.org