Introduction

Since its beginnings in the early sixties, admissible set theory has become a major source of interaction between model theory, resursion theory and set theory. In fact, for the student of admissible sets the old boundaries between fields disappear as notions merge, techniques complement one another, analogies become equivalences, and results in one field lead to results in another. This is the view of admissible sets we hope to share with the reader of this book.

Model theory, recursion theory and set theory all deal, in part, with problems of definability and set existence. *Definability theory* is (by definition) that part of mathematical logic which deals with such problems. The Craig Interpolation Theorem, Kleene's analysis of Δ_1^1 sets by means of the hyperarithmetic sets, Gödel's universe L of constructible sets and Shoenfield's Absoluteness Lemma are all major contributions to definability theory. The theory of admissible sets takes such apparently divergent results and makes them converge in a single coherent body of thought, one with ramifications for all parts of logic.

This book is written for the student who has taken a good first space year graduate course in logic. The specific material we presuppose can be summarized as follows. The student should understand the completeness, compactness and Löwenheim-Skolem theorems as well as the notion of elementary submodel. He should be familiar with the basic properties of recursive functions and recursively enumerable (hereinafter r.e.) sets. The student should have seen the development of intuitive set theory in some formal theory like ZF (Zermelo-Fraenkel set theory). His life will be more pleasant if he has some familiarity with the constructible sets before reading §§ II.5, 6 or V.4—8, but our treatment of constructible sets is self-contained.

A logical presentation of a reasonably advanced part of mathematics (which this book attempts to be) bears little relation to the historical development of that subject. This is particularly true of the theory of admissible sets with its complicated and rather sensitive history. On the other hand, a student is handicapped if he has no idea of the forces that figured in the development of his subject. Since the history of admissible sets is impossible to present here, we compromise by discussing how some of the older material fits into the current theory. We concentrate on those topics that are particularly relevant to this book. The prerequisites for understanding the introduction are rather greater than those for understanding the book itself.

Recursive ordinals and hyperarithmetic sets. In retrospect, the study of admissible ordinals began with the work of Church and Kleene on *notation systems and recursive ordinals* (Church-Kleene [1937], Church [1938], Kleene [1938].) This study began as a recursive counterpart to the classical theory of ordinals; *the least nonrecursive ordinal* ω_1^c is the recursive analogue of ω_1, the first uncountable ordinal. (Similarly for ω_2^c and ω_2, etc.) The theory of recursive ordinals had its most important application when Kleene [1955] used it in his study of the class of *hyperarithmetic sets*, the smallest reasonably closed class of sets of natural numbers which can be considered as given by the structure $\mathcal{N} = \langle \omega, +, \times \rangle$ of natural numbers. Kleene's theorem that

$$\text{hyperarithmetic} = \Delta_1^1$$

provided a construction process for the class of Δ_1^1 sets and constituted the first real breakthrough into (applied) second order logic. One of our aims is to provide a similar analysis for any structure $\mathfrak{M}$. Given $\mathfrak{M}$ we construct the smallest admissible set $\mathbb{HYP}_{\mathfrak{M}}$ above $\mathfrak{M}$ (in § II.5) and use it in the study of definability problems over $\mathfrak{M}$ (in Chapters IV and VI).

The study of hyperarithmetic sets generated a lot of discussion of the analogy between, on the one hand, the Π_1^1 and hyperarithmetic sets, and the r.e. and recursive sets on the other. These analogies became particularly striking when expressed in terms of representability in ω-logic and first order logic, by Grzegorczyk, Mostowski and Ryll-Nardzewski [1959]. The analogy had some defects, though, as the workers realized at the time. For example, the image of a hyperarithmetic function is hyperarithmetic, not just Π_1^1 as the analogy would suggest.

Kreisel [1961] analyzed this situation and discovered that the correct analogy is between Π_1^1 and hyperarithmetic on the one hand and r.e. and *finite* (not recursive) on the other. He went on to develop a recursion theory on the hyperarithmetic sets via a notation system. (He also proved the Kreisel Compactness Theorem for ω-logic: If a Π_1^1 theory T of second order arithmetic is inconsistent in ω-logic, then some hyperarithmetic subset $T_0 \subseteq T$ is inconsistent in ω-logic.) This theory was expanded in the *metarecursion theory* of Kreisel-Sacks [1965]. Here one sees how to develop, by means of an ordinal notation system, an attractive recursion theory on ω_1^c such that for $X \subseteq \omega$:

$$X \text{ is } \Pi_1^1 \text{ iff } X \text{ is } \omega_1^c\text{-r.e.,}$$

$$X \text{ is } \Delta_1^1 \text{ iff } X \text{ is } \omega_1^c\text{-finite.}$$

In § IV.3 we generalize this, by means of $\mathbb{HYP}_{\mathfrak{M}}$, to show that for any countable structure $\mathfrak{M}$ and any relation R on $\mathfrak{M}$:

$$R \text{ is } \Pi_1^1 \text{ on } \mathfrak{M} \text{ iff } R \text{ is } \mathbb{HYP}_{\mathfrak{M}}\text{-r.e.,}$$

$$R \text{ is } \Delta_1^1 \text{ on } \mathfrak{M} \text{ iff } R \text{ is } \mathbb{HYP}_{\mathfrak{M}}\text{-finite,}$$

thus providing a construction process for the Δ_1^1 relations over any countable structure $\mathfrak{M}$ whatsoever. The use of notation systems then allows us to transfer results from $\mathbb{HYP}_{\mathfrak{M}}$ to $\mathfrak{M}$ itself (see §§ V.5 and VI.5).

Constructible sets. The other single most important line of development leading to admissible sets also goes back to the late thirties. It began with the introduction by Gödel [1939] of the class L of constructible sets, in order to provide a model of set theory satisfying the axiom of choice and generalized continuum hypothesis (GCH).

Takeuti [1960, 1961] discovered that one could develop L by means of a recursion theory on the class Ord of all ordinals. He showed that Gödel's proof of the GCH in L corresponds to the following recursion theoretic stability: If κ is an uncountable cardinal and if $F\colon \mathrm{Ord} \to \mathrm{Ord}$ is ordinal-recursive then $F(\beta) < \kappa$ for all $\beta < \kappa$. In modern terminology, every uncountable cardinal is *stable*. Takeuti's definition of the ordinal-recursive functions was by means of schemata, Tagué [1964] provided an equivalent definition by means of an equation calculus obtained by adjoining an infinitistic rule to Kleene's equation calculus for ordinary recursion theory.

Admissible ordinals and admissible sets. The notion of admissible ordinal can be viewed as a common generalization of metarecursion theory and Takeuti's recursion theory on Ord. Kripke [1964] introduced admissible ordinals by means of an equation calculus. Platek [1965] gave an independent equivalent definition using schemata and another by means of machines as follows. Let α be an ordinal. Imagine an idealized computer capable of performing computations involving less than α steps. A function F computed by such a machine is called *α-recursive*. The ordinal α is said to be admissible if, for every α-recursive function F, whenever $\beta < \alpha$ and $F(\beta)$ is defined then $F(\beta) < \alpha$, that is, the initial segment determined by α is closed under F.

The first admissible ordinal is ω. An ordinal like $\omega + \omega$ cannot be admissible since, for $\alpha > \omega$, the equation

$$F(\beta) = \sup_{\gamma < \beta}(\omega + \gamma)$$

defines an α-recursive function and $F(\omega) = \omega + \omega$. The second admissible ordinal is, in fact, ω_1^c and the ω_1^c-recursion theory of Kripke and Platek agrees with that from metarecursion theory (see §§ IV.3 and V.5). The theorem of Takeuti mentioned above implies that every uncountable cardinal is admissible. The important advance made possible by the definition of admissible ordinal is that it allows one to study recursion theory on important ordinals (like ω_1^c) which are not cardinals.

Takeuti's work had shown that recursion theory on Ord amounts to definability theory on L. Analogously, the Kripke-Platek theory on an admissible ordinal α has a definability version on $L(\alpha)$, the sets constructible before stage α. It is this second approach which is most useful and is the one followed here. It leads us to consider *admissible sets*, sets $\mathbb{A}$ which, like $L(\alpha)$ for α admissible, satisfy *closure conditions* which insure a reasonable definability theory on $\mathbb{A}$. These principles are formalized in a first order set theory KP. In order to study general definability, though, not just definability theory in transitive sets, we must strengthen the general theory weakening KP to a new theory KPU. But this is taken up in detail at once in Chapter I.

Infinitary Logic. There is just one other idea that needs to be introduced here, that of infinitary logic. The model theory of $L_{\omega\omega}$, the usual first order predicate calculus, consists largely of global results, results which have to do with all models of some first order theory. These results have little to say about any one particular structure since only finite structures can be characterized up to isomorphism by a theory of $L_{\omega\omega}$. Recursion theory, on the other hand, is a local theory about the single structure $\mathcal{N}$. If we are to have a global theory with non-trivial local consequences, we must extend the model theory of $L_{\omega\omega}$ to stronger logics, logics which can characterize structures and properties not characterizable in $L_{\omega\omega}$. For the study of admissible sets the appropriate logics turn out to be admissible fragments $L_{\mathbb{A}}$ of $L_{\infty\omega}$, as developed in Barwise [1967, 1969a, b]. The countable case is studied in Chapter III; the uncountable case, in Chapters VII and VIII.

Some material not covered in this book. This book is a perspective on admissible sets, not a definitive treatment. It is far bigger and contains somewhat less material than we foresaw when we began writing. In particular, the following topics, all highly relevant to definability theory, are either omitted or slighted:

recursion theory in higher types,
Spector classes,
non-monotonic inductive definitions,
relative recursion theory on admissible ordinals,
forcing on admissible sets,
forcing and infinitary compactness arguments.

It is planned that some of these topics will be treated in other books in this series.

A note to the casual reader. There is one bit of notation that might be confusing to the casual reader of this book. We use $\mathfrak{A}$ for arbitrary models of the theory KPU or, more generally, for arbitrary structures for the language $L^* = L(\in, \ldots)$ in which KPU is formulated. We switch to the notation $\mathbb{A}$ when our structure is well founded. In 99.44% of the uses $\mathbb{A}$ will denote an admissible set.